

LUZERNE COUNTY MATHEMATICS CONTEST

Luzerne County Council of Teachers of Mathematics

Wilkes University - 2016 Junior Examination

(Section I)

NAME: _____

Address: _____

SCHOOL: _____

City/ZIP: _____

Telephone: _____

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) If $z = 1 - 2i$, find $\frac{1}{\bar{z}}$. (Express your answer in the form $a + bi$) 1) _____

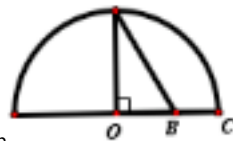
2) If $A = \{x \mid x - 1 > 0\}$ and $B = \{x \mid x \leq 3\}$, then $A \cap B =$ 2) _____
 (a) $(-1, 3)$ (b) $(1, 3]$ (c) $[1, 3)$ (d) $[-1, 3]$

3) What is the range of the function $f(x) = \begin{cases} \frac{1}{x}, & x > 1 \\ -x - 2, & x \leq 1 \end{cases}$? 3) _____

4) Find the quadratic function whose graph has vertex $(-1, 1)$ and passes through the point $(1, 9)$. (Express your answer in the form $f(x) = ax^2 + bx + c$) 4) _____

5) $y = \log_{\frac{1}{2}}(x^2 - 5x + 6)$ will be increasing in the interval 5) _____
 (a) $\left(\frac{5}{2}, \infty\right)$ (b) $(3, +\infty)$ (c) $\left(-\infty, \frac{5}{2}\right)$ (d) $(-\infty, 2)$

6) Given semicircle O with diameter of length 6 units and $m(\overline{BC}) = 1$, Find $m(\overline{AB})$. 6) _____



7) What is the value of $\sqrt{9} - 4\sin 30^\circ + (2016 - \pi)^0 - 2^2$ 7) _____

8) Three equal squares are joined as shown. What is $\tan(\alpha + \beta)$? 8) _____



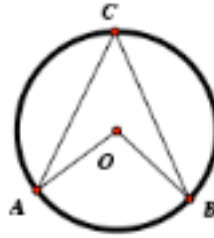
9) Find the sum of $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \dots + \frac{1}{99 \cdot 100}$. 9) _____

10) If $\tan\left(\frac{\pi}{4} - \alpha\right) = 3$ then what is the value of $c \tan \alpha$? 10) _____

(a) -2 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) 2

(OVER)

- 11) \overline{OA} and \overline{OB} are radii of circle O . C is on the circle. If $\overline{OA} \perp \overline{OB}$ then what is $m\angle ACB$?
- (a) 45° (b) 35° (c) 25° (d) 20°



11) _____

12) If $x = \sqrt{2} + 1$ then $\frac{x+1}{x} \div \left(x - \frac{1+x^2}{2x}\right) - 1 = ?$.

12) _____

13) Suppose $g(x) = \begin{cases} \arctan|x|, & x \leq 0 \\ \cos x, & x > 0 \end{cases}$. Find $g(g(-2))$.

13) _____

Express your answer only in terms of square roots of positive integers.

14) What is the smallest value attained by $f(x) = \sin x \cos x$?

14) _____

- (a) -1 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) 1

15) What rational number has a binary representation of $(0.\overline{01})_2 = (0.010101\dots)_2$?

15) _____

16) If the largest value of $x + my$ is $\frac{5}{3}$ and x and y satisfy

$$\begin{cases} y \geq x \\ y \leq 2x \\ x + y \leq 1 \end{cases}, \text{ then what is the value of } m?$$

16) _____

17) Express the solution to $4^x - 2^{x+1} - 8 < 0$ using interval notation.

17) _____

18) Let N denote a six-digit integer whose 6 digits are 1, 2, 3, 4, 5, and 6 in random order. What is the probability that N is divisible by 6?

18) _____

19) Solve for x : $(3x)^{\ln 3} = (5x)^{\ln 5}$

19) _____

20) How many trailing zeros will there be in $342!$?

20) _____

LUZERNE COUNTY MATHEMATICS CONTEST

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) Find the real part of $\frac{4 + 3i}{1 + 2i}$. 1) _____

2) Solve for x : $\begin{cases} 3x - 1 > 5 \\ 2(x + 2) < x + 7 \end{cases}$ 2) _____

3) In $\triangle ABC$ if $m(\overline{BC}) = \sqrt{5}$, $m(\overline{AC}) = 3$ and $\sin \angle C = 2 \sin \angle A$, What is $m(\overline{AB})$? 3) _____



4) If an even function $f(x)$ is increasing on $[0, \infty)$, then the range of x satisfying $f(2x - 1) < f\left(\frac{1}{3}\right)$ is 4) _____

- (a) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (b) $\left[\frac{1}{3}, \frac{2}{3}\right)$ (c) $\left(\frac{1}{2}, \frac{2}{3}\right)$ (d) $\left[\frac{1}{2}, \frac{2}{3}\right)$

5) Which of the following is divisible by 6?
(a) 10,100,101,022 (b) 10,100,101,012
(c) 10,100,201,022 (d) 11,100,201,022
(e) 11,101,201,022 5) _____

6) The aspect ratio of a certain screen is 2:1 (so the width of the screen is double its height). If the diagonal measures 20 in., what is the total area of the screen? 6) _____

7) Find the area contained within the ellipse $x^2 + 4y^2 - 2x + 16y + 1 = 0$ 7) _____

8) Find the exact value of $(1 - i)^{10}$. 8) _____

9) The geometric sequence $\{a_n\}_{n \geq 1}$ has a partial sum s_n . If $\frac{s_6}{s_3} = 3$, then $\frac{s_9}{s_6}$ is 9) _____
(a) 2 (b) $\frac{7}{3}$ (c) $\frac{8}{3}$ (d) 3

10) What is the value of the constant term in the expression $\left(x^2 + \frac{1}{x}\right)^6$? 10) _____

(OVER)

- 11) A line segment of length 2 is elevated at an angle θ from the positive x -axis. If a segment of length 1 is attached to the end, what is the maximum value of θ for which we get a triangle?



11) _____

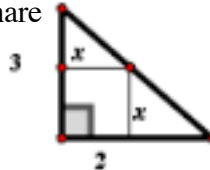
- 12) Which of the following functions has an inverse function equal to itself?

(a) $f(x) = x^2, x \geq 0$. (b) $f(x) = x^3, x \in \mathbb{R}$

(c) $f(x) = e^x, x \in \mathbb{R}$ (d) $f(x) = \frac{1}{x}, x > 0$

12) _____

- 13) A right triangle has legs of length 2 and 3. A square of side x is inscribed in the triangle so that the square and the triangle share the right angle. Find the perimeter of the square.



13) _____

- 14) We have a standard deck of cards consisting of 4 Jacks, 4 Queens, 4 Kings, and 4 Aces. If you are dealt 3 cards at random, what is the probability that your hand has at least 2 hearts?

14) _____

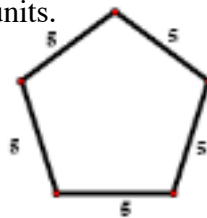
- 15) Find a 4th degree polynomial with integer coefficients having $\sqrt{2} + \sqrt{3}$ as a root.

15) _____

- 16) Find the area of a regular pentagon with side-length 5 units.

(a) $\frac{125}{4} \tan 54^\circ$ (b) $\frac{125}{4} \sin 54^\circ$

(c) $\frac{125}{4} \tan 72^\circ$ (d) $\frac{125}{4} \sin 72^\circ$



16) _____

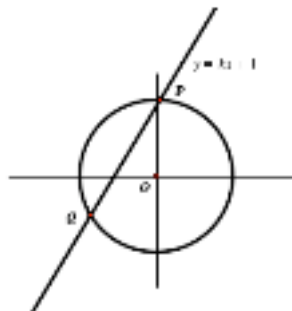
- 17) Five test scores have an average of 91, a median of 92 and a mode of 95. Find the sum of the lowest two test scores.

17) _____

- 18) Ten teams compete in the first round of a playoff tournament. How many different pairings are possible in the first round?

18) _____

- 19) A straight line, $y = kx + 1$ intersects a circle $x^2 + y^2 = 1$ at P and Q . If $m\angle POQ = 120^\circ$, what is the value of k ?



19) _____

- 20) Find the smallest possible base, b , such that the integer $(276)_b$ is divisible by 8.

20) _____

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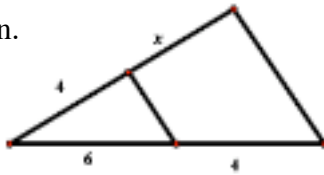
Telephone: _____

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) If $z_1 = 1 - i$, $z_1 \cdot z_2 = 1 + i$, then what is z_2 ?

1) _____

2) Find the value of x in the figure shown.



2) _____

3) If $a > 1$, $m = \log_a(a^2 + 1)$, $n = \log_a(a - 1)$, and $p = \log_a(2a)$, then

3) _____

(a) $n > m > p$

(b) $m > p > n$

(c) $m > n > p$

(d) $p > m > n$

4) In a sequence $\{a_n\}$, $a_k + a_1 = a_{k+1}$ and $a_1 = \frac{1}{9}$. What is a_{36} ?

4) _____

5) Find the domain of $f(x) = \ln(\arctan x - 1)$.

5) _____

6) For what values of k (if any) does the following system have infinitely many solutions ?

6) _____

$$\begin{cases} 5x + (k - 2)y = 2 \\ (2k)x + 6y = 4 \end{cases}$$

7) In a geometric sequence if $a_1=1$ and $a_4 = \frac{1}{8}$, then the partial sum

7) _____

s_{10} is (a) $2 - \frac{1}{2^8}$ (b) $2 - \frac{1}{2^9}$ (c) $2 - \frac{1}{2^{10}}$ (d) $2 - \frac{1}{2^{11}}$

8) Find all solutions to $\log_5(x - 2) + \log_5(x^2 + x - 7) = 1$.

8) _____

9) If $f_1(x) = x^{-\frac{1}{3}}$, $f_2(x) = x^{-1}$, and $f_3(x) = x^3$ then $f_1(f_2(f_3(2016))) = ?$

9) _____

10) $\sin 15^\circ \cos 75^\circ + \cos 15^\circ \sin 105^\circ =$

10) _____

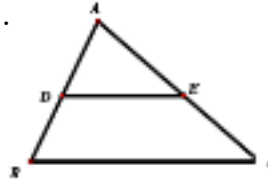
(a) 0 (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) 1

(OVER)

11) If the solution x to $\frac{x+k}{x+1} - \frac{k}{x-1} = 1$ is negative, what is the range of k ?

11) _____

12) D and E are on the sides of $\triangle ABC$ and $\overline{DE} \parallel \overline{BC}$. If $m\angle B = 60^\circ$, $m\angle AED = 40^\circ$, then $m\angle A =$
 (a) 100° (b) 90° (c) 80° (d) 70°



12) _____

13) Find all solutions to $\ln(\sin(x + |x|)) = 0$.

13) _____

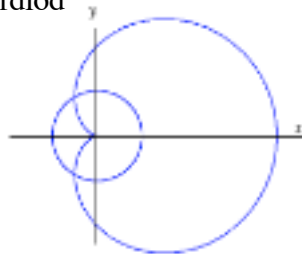
14) If a point $(3, 1)$ is on the curve of $f(x) = ax^2 - 2ax + b$, ($x \geq 1$), and the curve of $f^{-1}(x)$, the inverse function of $f(x)$, then

14) _____

- (a) $a = \frac{1}{2}, b = \frac{5}{2}$ (b) $a = \frac{1}{2}, b = -\frac{5}{2}$
 (c) $a = -\frac{1}{2}, b = \frac{5}{2}$ (d) $a = -\frac{1}{2}, b = -\frac{5}{2}$

15) Find all angles, $0 \leq \theta \leq 2\pi$, for which the cardioid $r = 2(1 + \cos \theta)$ intersects the unit circle.

15) _____



16) There are 12 balls in an urn labeled by the numbers 1, 2, ..., 11, 12. If two balls are taken from the urn at one time, what is the probability of getting at least one ball with an even number?

16) _____

17) Three positive integers multiply together to give 156. Two of the integers are even. Also, the sum of two of the integers is $\frac{5}{2}$ of the third. Find the three integers.

17) _____

18) What is the minimum value attained by $f(x) = \sqrt{3} \sin 2x - 2 \sin x$?

18) _____

19) Find the remainder when the polynomial $x^{101} + x^{51} + x^{33} + x^8 + x + 3$ is divided by $x^3 - x$.

19) _____

20) The following expression evaluates to what integer?

20) _____

$$\frac{\ln(2 \cos 15^\circ)}{\ln(2 \sin 15^\circ)}$$

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) If $M = \{-1, 1\}$ and $N = \left\{x \mid \frac{1}{2} < 2^{x+1} < 4\right\}$, then _____
 what is $M \cap N$?

2) $f(x) = \begin{cases} x^2 - 4x + 6, & x \geq 0 \\ x + 6, & x < 0 \end{cases}$. The solution to $f(x) > f(1)$ is _____
 (a) $(-3, 1) \cup (3, \infty)$ (b) $(-3, 1) \cup (2, \infty)$
 (c) $(-1, 1) \cup (3, \infty)$ (d) $(-\infty, -3) \cup (1, 3)$

3) What is the period of $f(x) = \sin 2x \cos x + \cos 2x \sin x$? _____

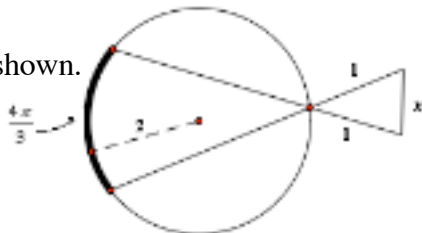
4) If $a^x = b^y = 3$ and $a + b = 2\sqrt{3}$ when $x, y \in \mathbb{R}, a > 1, b > 1$, _____
 what is the largest value of $\frac{1}{x} + \frac{1}{y}$?
 (a) 2 (b) $\frac{3}{2}$ (c) 1 (d) $\frac{1}{2}$

5) Find the solution to $|2x - 1| - x < 1$. _____

6) Find all x -values where the graph of $y = 2x^3 - 3x^2 - 2x + 2$ _____
 crosses the graph $y = x$. (*Hint:* All x -values are rational)

7) $f(x) = \begin{cases} x^2 + x, & x < 0 \\ -x^2, & x \geq 0 \end{cases}$. If $f(f(a)) \leq 2$, then what is then _____
 range of a ?

8) Find the length of x in the diagram shown. _____



9) Let n be any positive integer. What is the remainder when _____
 $n^2 + (n + 1)^2 + (n + 2)^2$ is divided by 3?

10) If $\sin \theta + \cos \theta = \frac{1}{5}$, and $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$, then what is _____
 $\cos 2\theta$?

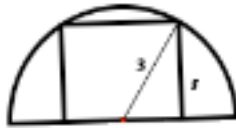
11) If $f(x) = 2\sin(\omega x + \varphi)$, $x \in \mathfrak{R}$, where $\omega > 0$, $|\varphi| < \frac{\pi}{2}$, 11) _____

has period π and $f(0) = 3$, then

- (a) $\omega = \frac{1}{2}$, $\varphi = \frac{\pi}{6}$ (b) $\omega = \frac{1}{2}$, $\varphi = \frac{\pi}{3}$
 (c) $\omega = 2$, $\varphi = \frac{\pi}{6}$ (d) $\omega = 2$, $\varphi = \frac{\pi}{3}$

12) On what interval will $\ln(x^2 + 1) - \ln(x + 3)$ be negative? 12) _____

13) A square is inscribed in a semicircle of radius 3. What is the length s of the square? 13) _____



14) There are 5 males and 4 females. Three people will be selected as a team. What is the probability the team contains both genders? 14) _____

15) If a sequence $\{a_n\}_{n \geq 1}$ has a partial sum $s_n = n^2 - 9n$, then $n = ?$ 15) _____

16) Find 2 two digit numbers (ie. a number where the tens digit is nonzero) that equals their ones digit squared plus 3 times their tens digit. 16) _____

17) If an unknown straight line and a straight line $x - 2y + 1 = 0$ are symmetric about $x = 1$, then the equation of the unknown straight line is 17) _____

- (a) $x + 2y - 1 = 0$ (b) $2x + y - 1 = 0$
 (c) $2x + y - 3 = 0$ (d) $x + 2y - 3 = 0$

18) For what values of x are the numbers $0, \ln(5^x + 1), \ln(5^x + 7)$ consecutive terms of an arithmetic sequence? 18) _____

19) If x and y , satisfy $\begin{cases} x + y \geq 3 \\ x - y \geq -1 \\ 2x - y \leq 3 \end{cases}$ then the smallest value of 19) _____

- $2x + 3y$ is
 (a) 6 (b) 7 (c) 8 (d) 23

20) Find the remainder when $\sum_{n=1}^{100} n! = 1! + 2! + 3! + \dots + 99! + 100!$ is divided by 14. 20) _____