LUZERNE COUNTY MATHEMATICS CONTEST

Luzerne County Council of Teachers of Mathematics
Wilkes University - 2016 Junior Examination
(Section I)

NAME: _____ **Address:** _____

SCHOOL: _____ **City/ZIP:** _____

Telephone: _____

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) If $z = 1 - 2i$, find $\frac{1}{z}$. (Express your answer in the form $a + bi$) **1)** $\frac{1}{5} - \frac{2}{5}i$

2) If $A = \{x \mid x - 1 > 0\}$ and $B = \{x \mid x \leq 3\}$, then $A \cap B =$ **2)** (b)

(a) $(-1, 3)$ **(b)** $(1, 3]$ **(c)** $[1, 3)$ **(d)** $[-1, 3]$

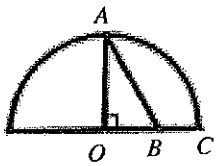
3) What is the range of the function $f(x) = \begin{cases} \frac{1}{x}, & x > 1 \\ -x - 2, & x \leq 1 \end{cases}$? **3)** $f(x) \geq -3$, or $[-3, \infty)$

4) Find the quadratic function whose graph has vertex $(-1, 1)$ and passes through the point $(1, 9)$. (Express your answer in the form $f(x) = ax^2 + bx + c$) **4)** $2x^2 + 4x + 3$

5) $y = \log_{\frac{1}{2}}(x^2 - 5x + 6)$ will be increasing in the interval **5)** d

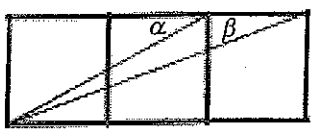
(a) $\left(\frac{5}{2}, \infty\right)$ **(b)** $(3, +\infty)$ **(c)** $\left(-\infty, \frac{5}{2}\right)$ **(d)** $(-\infty, 2)$

6) Given semicircle O with diameter of length 6 units and $m(\overline{BC}) = 1$, Find $m(\overline{AB})$. **6)** $\sqrt{13}$



7) What is the value of $\sqrt{9} - 4\sin 30^\circ + (2016 - \pi)^0 - 2^2$? **7)** -2

8) Three equal squares are joined as shown. What is $\tan(\alpha + \beta)$? **8)** 1



9) If $x = \sqrt{2} + 1$ then $\frac{x+1}{x} = ?$ **9)** $\sqrt{2}$

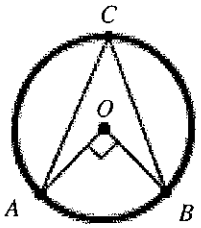
10) Which of the following numbers is definitely irrational? **10)** (b)

(a) $\log_3 81$ **(b)** $\sqrt{\frac{\pi}{2} - \frac{3}{4}}$ **(c)** $e^{\ln 7}$ **(d)** $4^{\frac{\ln 3}{\ln 4}}$

(OVER)

11) $\overline{OA}$ and $\overline{OB}$ are radii of circle O. C is on the circle. If $\overline{OA} \perp \overline{OB}$ then what is $m\angle ACB$?

(a) 45° **(b)** 35° **(c)** 25° **(d)** 20°



11) (a)

12) Find all solutions to the system $\begin{cases} 2x - 3y = 3 \\ x + 5y = 8 \end{cases}$

12) $x = 3;\ y = 1$

13) Suppose $g(x) = \begin{cases} \arctan|x|, & x \le 0 \\ \cos x, & x > 0 \end{cases}$. Find $g(g(-2))$.

Express your answer only in terms of square roots of integers.

13) $\frac{\sqrt{5}}{5}$

14) What is the smallest value attained by $f(x) = \sin x \cos x$?

(a) -1 **(b)** $-\frac{1}{2}$ **(c)** $\frac{1}{2}$ **(d)** 1

14) (b)

15) What rational number has a binary representation of $(0.\overline{01})_2 = (0.010101\ldots)_2$?

15) $\frac{1}{3}$

16) The region enclosed by $\begin{cases} y \ge x \\ y \le 2x \\ x + y \le 1 \end{cases}$ has longest bounding curve of what length?

16) $\frac{\sqrt{5}}{3}$

17) Express the solution set to $4^x - 2^{x+1} - 8 < 0$ using interval notation.

17) $(-\infty, 2)$

18) Find the sum of $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{99\cdot 100}$.

18) $\frac{99}{100}$

19) The equation $(3x)^{\ln 3} = (5x)^{\ln 5}$ has what rational number for a solution?

19) $\frac{1}{15}$

20) We have two 12-sided dice with the numbers 1 through 12 on the faces. What is the probability of rolling 2 even numbers?

20) $\frac{1}{4}$

LUZERNE COUNTY MATHEMATICS CONTEST

Luzerne County Council of Teachers of Mathematics
Wilkes University - 2016 Senior Examination
(Section I)

NAME: _____ **Address:** _____

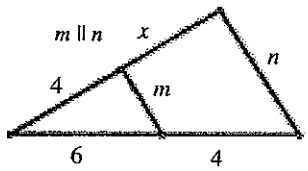
SCHOOL: _____ **City/ZIP:** _____

Telephone: _____

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) If $z_1 = 1 - i$, $z_1 \cdot z_2 = 1 + i$, then what is z_2 ? Express your answer in the form $a + bi$. **1)** i

2) Find the value of x in the figure shown. **2)** $\frac{8}{3}$ or $2\frac{2}{3}$



3) If $a > 1$, $m = \log_a(a^2 + 1)$, $n = \log_a(a - 1)$, and $p = \log_a(2a)$, then **3)** (b)

(a) $n > m > p$ **(b)** $m > p > n$
(c) $m > n > p$ **(d)** $p > m > n$

4) In a sequence $\{a_n\}$, $a_k + a_1 = a_{k+1}$ and $a_1 = \frac{1}{9}$. What is a_{36} ? **4)** 4

5) Find the domain of $f(x) = \ln\left(\arctan x - \frac{\pi}{4}\right)$. **5)** $x > 1$ or $(1, \infty)$

6) $f(x) = \begin{cases} x^2 + x, & x < 0 \\ -x^2, & x \geq 0 \end{cases}$. If $f(f(a)) \leq 2$, then what is then range of a ? **6)** $a \leq \sqrt{2}$ or $(-\infty, \sqrt{2})$

7) In a geometric sequence, if $a_1 = 1$ and $a_4 = \frac{1}{8}$, then the partial sum s_{10} is **(a)** $2 - \frac{1}{2^8}$ **(b)** $2 - \frac{1}{2^9}$ **(c)** $2 - \frac{1}{2^{10}}$ **(d)** $2 - \frac{1}{2^{11}}$ **7)** (b)

8) Find all real solutions to $\log_5(x - 2) + \log_5(x^2 + x - 7) = 1$. **8)** 3

9) If $f_1(x) = x^{-\frac{1}{3}}$, $f_2(x) = x^{-1}$, and $f_3(x) = x^3$ then $f_1(f_2(f_3(2016))) = ?$ **9)** 2016

10) $\sin 15^\circ \cos 75^\circ + \cos 15^\circ \sin 105^\circ =$ **10)** (d)

(a) 0 **(b)** $\frac{1}{2}$ **(c)** $\frac{\sqrt{3}}{2}$ **(d)** 1

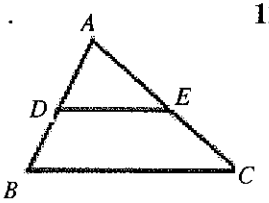
(OVER)

11) If the solution x to $\dfrac{x+k}{x+1} - \dfrac{k}{x-1} = 1$ is negative, what is the range of k?

11) $k > \frac{1}{2}$ or $(\frac{1}{2}, \infty)$

12) D and E are on the sides of $\triangle ABC$ and $\overline{DE} \parallel \overline{BC}$. If $m\angle B = 60^\circ$, $m\angle AED = 40^\circ$, then $m\angle A =$

(a) 100° **(b)** 90° **(c)** 80° **(d)** 70°



12) (C)

13) Find all real solutions to $\ln\left(\sin\left(x + \lfloor x \rfloor\right)\right) = 0$ that lie in the range $-4\pi \le x \le 4\pi$.

13) $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

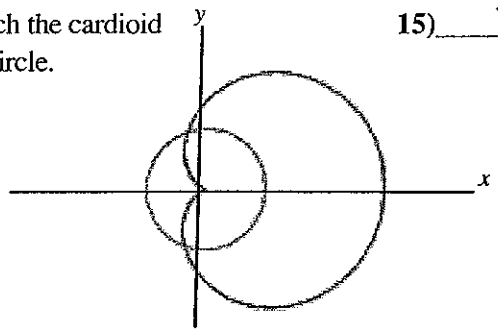
14) If a point $(3, 1)$ is on the curve of $f(x) = ax^2 - 2ax + b$, $(x \ge 1)$, and the curve of $f^{-1}(x)$, the inverse function of $f(x)$, then

(a) $a = \dfrac{1}{2}, b = \dfrac{5}{2}$ **(b)** $a = \dfrac{1}{2}, b = -\dfrac{5}{2}$

(c) $a = -\dfrac{1}{2}, b = \dfrac{5}{2}$ **(d)** $a = -\dfrac{1}{2}, b = -\dfrac{5}{2}$

14) (C)

15) Find all angles, $0 \le \theta \le 2\pi$, for which the cardioid $r = 2(1 + \cos\theta)$ intersects the unit circle.



15) $\frac{2\pi}{3}, \frac{4\pi}{3}$

16) There are 12 balls in an urn labeled by the numbers 1, 2, ..., 11, 12. If two balls are taken from the urn at one time, what is the probability of getting at least one ball with an even number?

16) $\frac{51}{66}$

17) Three positive integers multiply together to give 156. Two of the integers are even. Also, the sum of two of the integers is $\dfrac{5}{2}$ of the third. Find the three integers.

17) 2, 6, 13

18) If $\cos\alpha = \dfrac{4}{5}$, then what is $\cos 2\alpha$?

18) $\frac{7}{25}$

19) Find the remainder when the polynomial $x^{101} + x^{51} + x^{33} + x^8 + x + 3$ is divided by $x^3 - x$.

19) $x^2 + 4x + 3$ or $(x+1)(x+3)$

20) What is the minimum value attained by $f(x) = \sqrt{3}\sin 2x - 2\sin^2 x$?

20) -3

LUZERNE COUNTY MATHEMATICS CONTEST

Luzerne County Council of Teachers of Mathematics
Wilkes University - 2016 Senior Examination
(Section II)

NAME: _____ **Address:** _____

SCHOOL: _____ **City/ZIP:** _____

Telephone: _____

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) If $M = \{-1, 1\}$ and $N = \left\{x \mid \frac{1}{2} < 2^{x+1} < 4\right\}$, then what is $M \cap N$?

1) $\{-1\}$

2) If $f(x) = \begin{cases} x^2 - 4x + 6, & x \geq 0 \\ x + 6, & x < 0 \end{cases}$, then all real solutions to $f(x) > f(1)$ are

(a) $(-3, 1) \cup (3, \infty)$ **(b)** $(-3, 1) \cup (2, \infty)$
(c) $(-1, 1) \cup (3, \infty)$ **(d)** $(-\infty, -3) \cup (1, 3)$

2) (a)

3) What is the period of $f(x) = \sin 2x \cos x + \cos 2x \sin x$?

3) $\frac{2\pi}{3}$

4) If $a^x = b^y = 3$ and $a + b = 2\sqrt{3}$ when $x, y \in \mathbb{R}$, $a > 1, b > 1$, what is the largest value of $\frac{1}{x} + \frac{1}{y}$?

(a) 2 **(b)** $\frac{3}{2}$ **(c)** 1 **(d)** $\frac{1}{2}$

4) (c)

5) Find all real solutions to $|2x - 1| - x < 1$.

5) $0 < x < 2$ or $(0, 2)$

6) Find all x-values where the graph of $y = 2x^3 - 3x^2 - 2x + 2$ crosses the graph $y = x$. (*Hint*: All values of x are rational)

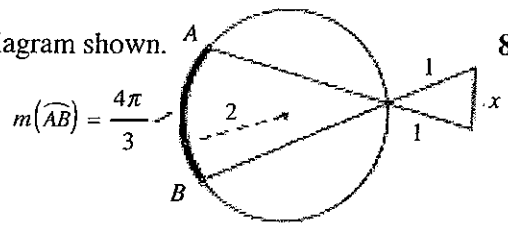
6) $-1, \frac{1}{2}, 2$

7) For what values of k (if any) does the following system have infinitely many solutions ?

$$\begin{cases} 5x + (k-2)y = 2 \\ (2k)x + 6y = 4 \end{cases}$$

7) 5

8) Find the length of of x in the diagram shown.



8) 1

9) Let n be any positive integer. What is the remainder when $n^2 + (n+1)^2 + (n+2)^2$ is divided by 3?

9) 2

10) What is the value of $\sin 75^\circ$?

10) $\frac{\sqrt{2}(1+\sqrt{3})}{4}$ or $\frac{\sqrt{2+\sqrt{3}}}{2}$

(OVER)

11) If $f(x) = 2\sin(\omega x + \varphi)$, $x \in \mathbb{R}$, where $\omega > 0$, $|\varphi| < \frac{\pi}{2}$, has period π and $f(0) = 1$, then

(a) $\omega = \frac{1}{2}, \varphi = \frac{\pi}{6}$ **(b)** $\omega = \frac{1}{2}, \varphi = \frac{\pi}{3}$

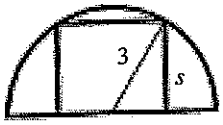
(c) $\omega = 2, \varphi = \frac{\pi}{6}$ **(d)** $\omega = 2, \varphi = \frac{\pi}{3}$

11) (C)

12) On what interval will $\ln(x^2 + 1) - \ln(x + 3)$ be negative?

12) $(-1, 2)$ or $-1 < x < 2$

13) A square is inscribed in a semicircle of radius 3. What is the length s of the square ?



13) $\frac{6\sqrt{5}}{5}$

14) There are 5 males and 4 females. Three people will be selected as a team. What is the probability that the team contains both genders?

14) $\frac{5}{6}$

15) If a sequence $\{a_n\}_{n\geq 1}$ has a partial sum $s_n = n^2 - 9n$, then $a_n = ?$

15) $2n - 10$

16) What is the remainder when $6x^5 + 3x^4 + 9x - 2$ is divided by $x - 1$?

16) 16

17) If an unknown straight line and a straight line $x - 2y + 1 = 0$ are symmetric about $x = 1$, then the equation of the unknown straight line is

(a) $x + 2y - 1 = 0$ **(b)** $2x + y - 1 = 0$

(c) $2x + y - 3 = 0$ **(d)** $x + 2y - 3 = 0$

17) (d)

18) For what values of x are the numbers 0, $\ln(5^x + 1)$, $\ln(5^x + 7)$ consecutive terms of an arithmetic sequence?

18) $\log_5 2$ or $\frac{\ln 2}{\ln 5}$

19) Find all real solutions to $\log_2(x + 1) - \log_2(x + 2) > 3$.

19) ϕ or no solution

20) If x and y satisfy $\begin{cases} x + y \geq 3 \\ x - y \geq -1 \\ 2x - y \leq 3 \end{cases}$ then the smallest value of $2x + 3y$ is

(a) 6 **(b)** 7 **(c)** 8 **(d)** 23

20) (b)

LUZERNE COUNTY MATHEMATICS CONTEST

Luzerne County Council of Teachers of Mathematics
Wilkes University - 2016 Junior Examination
(Section II)

NAME: _____ **Address:** _____

SCHOOL: _____ **City/ZIP:** _____

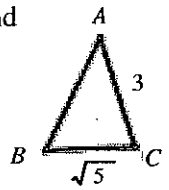
Telephone: _____

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) Find the real part of $\frac{4+3i}{1+2i}$. **1)** 2

2) Find all real solutions to the system of equations $\begin{cases} 3x-1>5 \\ 2(x+2)<x+7 \end{cases}$ **2)** $2<x<3$ or $(2,3)$

3) In $\triangle ABC$ if $m(\overline{BC}) = \sqrt{5}$, $m(\overline{AC}) = 3$ and $\sin\angle C = 2\sin\angle A$, What is $m(\overline{AB})$? **3)** $2\sqrt{5}$



4) If an even function $f(x)$ is increasing on $[0, \infty)$, then the range of x satisfying $f(2x-1) < f\left(\frac{1}{3}\right)$ is **4)** (a)

(a) $\left(\frac{1}{3}, \frac{2}{3}\right)$ **(b)** $\left[\frac{1}{3}, \frac{2}{3}\right)$ **(c)** $\left(\frac{1}{2}, \frac{2}{3}\right)$ **(d)** $\left[\frac{1}{2}, \frac{2}{3}\right)$

5) Which of the following is divisible by 6? **5)** (C)

(a) 10,100,101,022 **(b)** 10,100,101,012
(c) 10,100,201,022 **(d)** 11,100,201,022

6) The aspect ratio of a certain rectangular screen is 2:1 (so the width of the screen is double its height). If the diagonal measures 20 in., what is the total area of the screen? **6)** $160\ in^2$

7) Find the area contained within the ellipse $x^2 + 4y^2 - 2x + 16y + 1 = 0$ **7)** 8π

8) Find the exact value of $(1-i)^{10}$. **8)** $-32i$

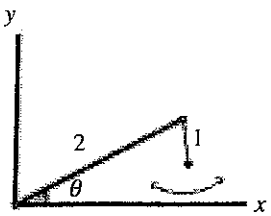
9) A committee is to be comprised of 1 chairperson and 2 regular members. How many different committees can be formed from a group of 6 people? **9)** (C)

a) 20 **(b)** 30 **(c)** 60 **(d)** 120

10) What is the value of the constant term in the expression $\left(x^2 + \frac{1}{x}\right)^6$? **10)** 15

(OVER)

11) A line segment of length 2 is elevated at an angle θ from the positive x-axis. If a line segment of length 1 is attached to the end, what is the maximum value of θ, in degrees, for which we get a triangle?



11) $\frac{\pi}{6}$ or 30°

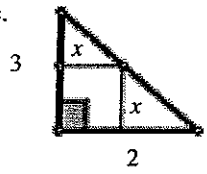
12) Which of the following functions has an inverse function equal to itself?

(a) $f(x) = x^2$, $x \geq 0$. **(b)** $f(x) = x^3$, $x \in \mathbb{R}$

(c) $f(x) = e^x$, $x \in \mathbb{R}$ **(d)** $f(x) = \frac{1}{x}$, $x > 0$

12) (d)

13) A right triangle has legs of length 2 and 3. A square of side x is inscribed in the triangle so that the square and the triangle share the right angle. Find the perimeter of the square.



13) $\frac{24}{5}$ or $4\frac{4}{5}$

14) Three cards are dealt at random from a standard deck of playing cards, what is the probability that your hand has at least 2 hearts?

14) $\frac{832}{5525}$

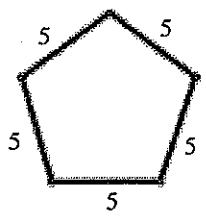
15) The average of 5 numbers is 21. If the average of the first 4 is 20, what is the fifth number?

15) 25

16) Find the area of a regular pentagon with side-length 5 units.

(a) $\frac{125}{4}\tan 54^\circ$ **(b)** $\frac{125}{4}\sin 54^\circ$.

(c) $\frac{125}{4}\tan 72^\circ$. **(d)** $\frac{125}{4}\sin 72^\circ$



16) $\frac{125}{4}\tan 54^\circ$

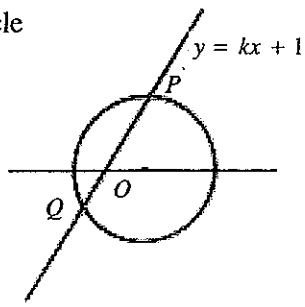
17) A geometric sequence of strictly positive terms has first term 2 and the sum of the first 3 terms is 26. What is the common ratio?

17) 3

18) What rational number is $(27)^{-\frac{4}{3}}$?

18) $\frac{1}{81}$

19) A straight line, $y = kx + 1$ intersects a circle $x^2 + y^2 = 1$ at P and Q. If $m\angle POQ = 120^\circ$, what is the value of k?



19) $\sqrt{3}$

20) Find all real roots of $x^4 + x^2 - 12 = 0$.

20) $\pm\sqrt{3}$