Luzerne County Mathematics Contest

Luzerne County Council of Teachers of Mathematics
Wilkes University – 2017 Junior Examination
(Section 1)

NAME:____________________________ **ADDRESS:**____________________

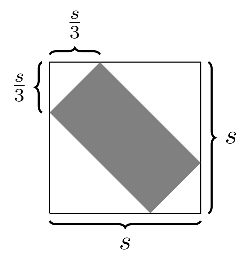
SCHOOL:____________________________ **CITY/ZIP:**____________________

TELEPHONE:____________________

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) If $z_1 = 1 - i$ and $z_1 \cdot z_2 = 1 + i$, find z_2. Express your answer in the standard form $a + bi$. **1)** ____________

2) A rectangle is inscribed in a square as shown. Find the ratio of the inscribed rectangle's area to the area of the outer square. **2)** ____________



3) Find $f(f(2))$ for the function **3)** ____________

$$f(x) = \begin{cases} \frac{1}{x}, & x > 1 \\ -x - 2, & x \le 1 \end{cases}.$$

4) How much does the maximum value of $f(x) = 2 - 2\sin(x)$ exceed the maximum value of $g(x) = 2 - (2 - x)^2$? **4)** ____________

5) For how many values of x do the curves $y = x^2$ and $y = 2^x$ intersect? **5)** ____________

6) If $\sin\alpha = \sqrt{5}/5$, then $\sin^4\alpha - \cos^4\alpha$ is equal to **6)** ____________

(a) $-1/5$ (b) $-3/5$
(c) $1/5$ (d) $3/5$

7) Find all real solutions to the inequality $3x + 5|x| < 16$. Express your answer in interval notation. **7)** ____________

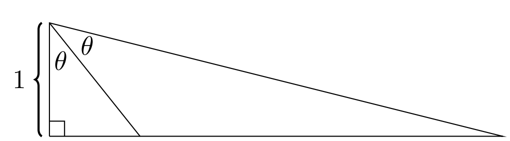
8) If $x^2 + (2m+1)x + m^2 - 1 = 0$ has two distinct real roots, then the parameter m must lie in what interval? Express your answer in interval notation. **8)** ____________

9) California has approximately 25,000,000 adults. What is the minimum number of 3-digit area codes needed to ensure that each adult can have a distinct 10-digit phone number. Assume that phone numbers are of the form **9)** ____________

$$\underbrace{NXX}_{\text{Area}} - NXX - XXXX$$

where N is any digit from 2 to 9 and X is any digit from 0 to 9.

10) The scalene right triangle below has the property that the angle bisector shown splits the triangle into two smaller triangles, one of which has three times the area of the other. **10)** ____________



The angle θ indicated above must be equal to

(a) $\arctan\left(\sqrt{2}/2\right)$ (b) $\arctan\left(\sqrt{3}/2\right)$
(c) $\arctan(1/3)$ (d) $\arctan(1/2)$

(OVER)

11) Find all real solutions to $\sqrt{x} = \sqrt[3]{4x+25}$. **11)** ______________

12) Find all real solutions to the equation **12)** ______________

$$\ln\left(\ln\left(2 + \log_2(x+1)\right)\right) = 0.$$

13) If the sum of coefficients in the expansion of

$$\left(x + \frac{1}{x}\right)^n$$

is 64, then the constant term must be **13)** ______________

(a) 10 (b) 20
(c) 30 (d) 120

14) A mother has two children. Find the age of the mother based on the following clues.

1) All ages are integers.
2) The mother is at least 20 years old but no older than 60.
3) Both children are at least 2 years old.
4) The mother is at least 6 times older than the eldest child.
5) The product of all three ages is 390.

14) ______________

15) Find the positive integer that is equal to the following expression. **15)** ______________

$$\log_3(4) \cdot \log_4(5) \cdot \log_5(6) \cdots \log_{80}(81)$$

16) The three sides of a triangle with an angle of 120° are in arithmetic progression with a difference of 2. Find the lengths of all three sides and list them in order. **16)** ______________

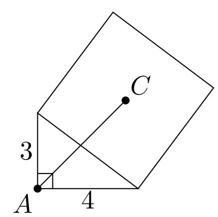
17) Which of the following is the lower bound of the range of

$$f(x) = \frac{1}{1 - \arctan(x)}$$

for values of x less than 0? **17)** ______________

(a) $1/(\pi+1)$ (b) $1/(\pi-1)$
(c) $1/\left(\frac{\pi}{2}-1\right)$ (d) $1/\left(\frac{\pi}{2}+1\right)$

18) The hypotenuse of a right triangle with sides 3, 4, and 5 is made into a square as shown to the right. Find the length of segment AC where C is the center of the square.



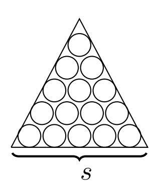
18) ______________

19) Find the minimum value of $z = x + y$ if x and y satisfy

$$\begin{cases} 2x + y \geq 4 \\ x - y \geq -1 \\ x - 2y \leq 2 \end{cases}.$$

19) ______________

20) Fifteen billiard balls with diameter 9/4 inches are arranged on a flat surface so that they form an equilateral triangle. What is the side-length, s, of the smallest equilateral triangle which is large enough to enclose all fifteen billiard balls in its interior?



20) ______________

Luzerne County Mathematics Contest

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) If $A = \{x|x-1>0\}$ and $B = \{x|x \leq 3\}$, then $A \cap B =$

(a) $(-1, 3)$ (b) $(1, 3]$
(c) $[1, 3)$ (d) $[-1, 3]$

1) ____________

2) Find all strictly positive solutions to

$$4^{x^3-2} = 16^{4x-1}.$$

2) ____________

3) Suppose that f is an odd function. Find $f(-2)-f(-3)$ given that $f(3)-f(2)=1$.

3) ____________

4) How many triangles have an angle of 60° with an adjacent side of length 4 and an opposite side of length 7/2?

4) ____________

5) If point $(a, 9)$ is on the graph of $y = 3^x$, compute $\tan\left(\frac{a\pi}{6}\right)$.

5) ____________

6) If x_0 is the solution to $\left(\frac{1}{2}\right)^x = x^{1/3}$, then x_0 is in the interval

(a) $\left(\frac{2}{3}, 1\right)$ (b) $\left(\frac{1}{2}, \frac{2}{3}\right)$

(c) $\left(\frac{1}{3}, \frac{1}{2}\right)$ (d) $\left(0, \frac{1}{3}\right)$

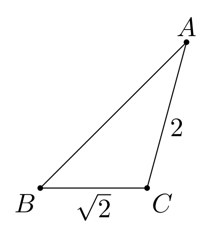
6) ____________

7) The domain of $f(x) = \sqrt{x+3} + \log_2(6-x)$ is

(a) $\{x|x>6\}$ (b) $\{x|-3<x<6\}$
(c) $\{x|x>-3\}$ (d) $\{x|-3 \leq x < 6\}$

7) ____________

8) In $\triangle ABC$ shown below, if the length of segment BC is $\sqrt{2}$, the length of segment AC is 2, and $\sin(m\angle B) + \cos(m\angle B) = \sqrt{2}$, find $m\angle A$, the measure of angle A (in radians).



8) ____________

9) Find the sum of the series

$$4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \frac{4}{81} + \cdots$$

9) ____________

10) Find all solutions to the non-linear system of equations

$$\begin{cases} 2x^2 - 4xy + 3y^2 = 36 \\ 3x^2 - 4xy + 2y^2 = 36 \end{cases}$$

which lie in Quadrant II. Give your answers as ordered pairs.

10) ____________

(OVER)

11) Find the smallest positive integer greater than 20 which has a remainder of 2 when divided by 3 and a remainder of 3 when divided by 7. **11)** ________

12) Find the minimum value of the function **12)** ________

$$f(x) = \sqrt{2x^2 - 4x + 7}.$$

13) Find all solutions to **13)** ________

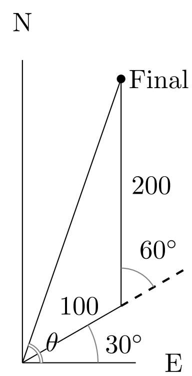
$$\left(\sqrt{2+\sqrt{3}}\right)^x + \left(\sqrt{2-\sqrt{3}}\right)^x = 4$$

14) If $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{1}{2}$ and $\frac{\pi}{2} \le \theta \le \pi$, compute $\sin\theta + \cos\theta$. **14)** ________

15) If $f(x) = a + bx$, what are the real values of a and b such that the following conditions hold? **15)** ________

$$\begin{aligned} f(f(f(0))) &= 2 \\ f(f(f(1))) &= 29 \end{aligned}$$

16) A plane flies 100 miles from home base on a bearing of 30° north of due east. The plane then turns 60° toward north and continues flying for 200 miles. Find $\tan(\theta)$ where θ is the bearing (measured north of east) from home base to the final location of the plane. **16)** ________



17) Suppose a and b are integers and that $a \neq 0$. If $a + b < 4$ and the quadratic polynomial **17)** ________

$$f(x) = ax^2 + 4x + b$$

has no real roots, what is the largest possible value of b?

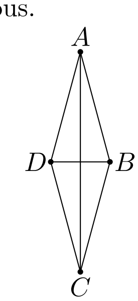
18) Find $\sin(4\theta)$ if $\sin(\theta) = \frac{4}{5}$ and $\frac{\pi}{2} < \theta < \pi$. **18)** ________

19) Let n be an integer greater than 3. For what n is the number **19)** ________

$$(0.\overline{13})_n = 0.131313\ldots \text{ (base } n)$$

the reciprocal of a positive integer?

20) The side-length, ℓ, of a rhombus is equal to the geometric mean of the lengths of its two diagonals. Find $m\angle B$, the angle measure (in degrees) of the obtuse angle in the rhombus. **20)** ________



$$\begin{aligned} \ell &= m(AB) = m(BC) = m(CD) = m(AD) \\ d_1 &= m(AC) \\ d_2 &= m(BD) \\ \ell &= \sqrt{d_1 d_2} \end{aligned}$$

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) True or False: An irrational number raised to an irrational power is always an irrational number.

1) ________________

2) Write the following complex number in the standard form $a + bi$.

$$\frac{10i}{3+i}$$

2) ________________

3) Simplify the expression

$$(3-\pi)^0 + 4\sin\left(\frac{\pi}{4}\right) - \sqrt{8} + \left|1-\sqrt{3}\right|.$$

3) ________________

4) Find all real solutions to the equation

$$\ln(x) + \ln(x+2) = \ln(x+6).$$

4) ________________

5) If $a + b = 2$, then which of the following choices is equal to

$$\left(a - \frac{b^2}{a}\right)\left(\frac{a}{a-b}\right).$$

(a) 2 (b) -2
(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

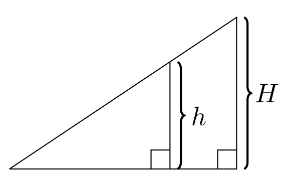
5) ________________

6) Find all real solutions to the system of equations below. List your answers as ordered pairs (x, y).

$$\begin{cases} x^2 - xy + 1 &= 0 \\ y &= 2x^2 \end{cases}$$

6) ________________

7) The smaller right triangle shown below has height h and exactly one-half the area of the larger right triangle (with height H). Find the exact value of the ratio $\frac{h}{H}$.



7) ________________

8) Suppose January 1 is on a Monday in a non-leap year. Tax Day, April 15, is 104 days later. What day of the week will Tax Day fall on?

8) ________________

9) Find all real solutions to the equation

$$\sqrt{x+3} - \sqrt{x-2} = \sqrt{6x-11}.$$

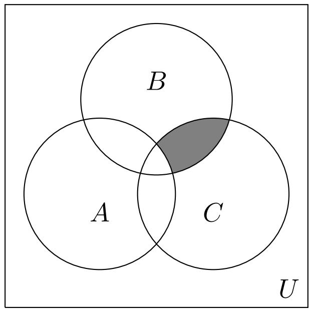
9) ________________

10) In an arithmetic sequence $\{a_n\}_{n\geq 1}$, if $a_3 = 6$ and the sum of the first 20 terms is 20, what is the sum of the first 10 terms?

10) ________________

(OVER)

11) Which of the following sets describes the shaded region in the Venn diagram below?



a) $[A^c \cup B^c \cup C^c]^c$

b) $[A \cup B^c \cup C^c]^c$

c) $[A^c \cup B \cup C^c]^c$

d) $[A^c \cup B^c \cup C]^c$

(S^c stands for the complement of the set S relative to the universal set U.)

11) ______________

12) Find p, q, and r with $r < 0$ so that

$$4x^4 - 12x^3 + x^2 + 12x + 4 = \left(px^2 + qx + r\right)^2$$

is valid for all x. List your answer as a triple (p, q, r).

12) ______________

13) Find the positive integer which is equal to the nested radical expression

$$\sqrt[3]{24 + \sqrt[3]{24 + \sqrt[3]{24 + \cdots}}}.$$

13) ______________

14) Find the minimum of the quantity $z = 2x + 3y$ given that x and y satisfy the following conditions.

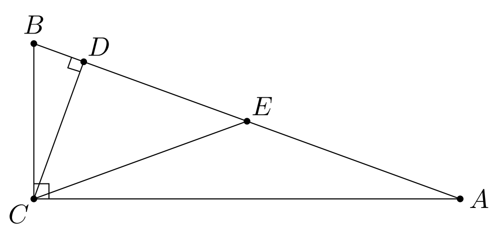
$$\begin{cases} x + y & \geq 2 \\ 2x - y & \leq 4 \\ x - y & \geq 0 \end{cases}$$

14) ______________

15) If $\sin\theta + \cos\theta = \frac{1}{5}$ and $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$, compute $\cos 2\theta$.

15) ______________

16) In right triangle $\triangle ABC$, segment CD is the altitude to the hypotenuse AB. Segment CE is the median to AB. If the measure of angle B is $70°$, what is the measure of $\angle DCE$ (in degrees)?



16) ______________

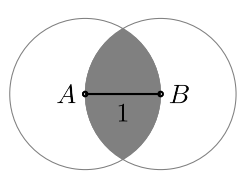
17) What is the largest prime divisor of every 3-digit number with three identical non-zero digits?

17) ______________

18) Two integers are drawn from the set $\{1, 2, 3, \ldots n\}$ without replacement. Determine n given that the probability that the sum of the two integers is 5 equals $\frac{1}{14}$.

18) ______________

19) Let points A and B be the centers of two circles with radii equal to 1. Assume that A and B are 1 unit apart. Find the area of the intersection of these two circles.



19) ______________

20) Suppose a and b are positive integers satisfying

$$\begin{aligned} ab &= a^b \\ \frac{a}{b} &= a^{2b}. \end{aligned}$$

Find $8a + 3b$.

20) ______________

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) If set $A = \{x|x$ is an integer divisible by 2 or 3$\}$ and set $B = \{x| -2x^2 + 21x - 10 > 0\}$, find $|A \cap B|$ (the cardinality of the intersection). **1)** __________

2) Find $f(-8)$ for the function **2)** __________

$$f(x) = \begin{cases} 2^x, & x > 0 \\ f(x+3), & x \leq 0 \end{cases}.$$

3) Find x in the 4×4 sudoku puzzle below. Remember that each row, column, and 2×2 corner block must have each of the digits $1, 2, 3$, and 4 appearing exactly once! **3)** __________

		2	
1			
x	4		
			3

4) Find the exact value of **4)** __________

$$\tan\left[\arcsin\left(-\frac{3}{4}\right)\right].$$

5) Find all real solutions to the following inequality. Express your answer in interval notation. **5)** __________

$$|2x - 1| - x < 1$$

6) A professor uses the function **6)** __________

$$s(x) = \frac{x + 10\sqrt{x}}{2}$$

to curve scores on an exam. What minimum raw score is needed to ensure that the curved score is 72 or higher?

7) Find the domain of the following function. Give your answer as a union of increasing intervals (in interval notation). **7)** __________

$$f(x) = \sqrt{\frac{x^2 - 5x}{x^2 - 9}}.$$

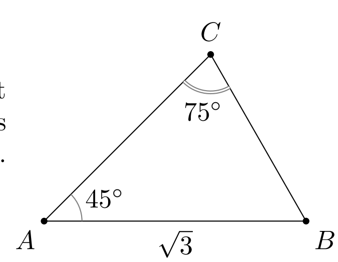
8) Find all real solutions to **8)** __________

$$\tan^2\theta = \tan\theta$$

in the interval $0 \leq \theta \leq 2\pi$. List your answers in increasing order.

9) How many distinct anagrams of the word LLAMAS can you make? (An anagram does not have to be an actual word in the English language.) **9)** __________

10) In $\triangle ABC$, the length of segment AB is $\sqrt{3}$, the measure of angle A is $45°$, and the measure of angle C is $75°$. Find the length of segment BC. **10)** __________



(OVER)

11) A cubic polynomial has a root of multiplicity 2 at $x = 2$. The graph of the polynomial passes through the points $(1, 1)$ and $(3, -3)$. Find the remaining root of the polynomial. **11)** ________

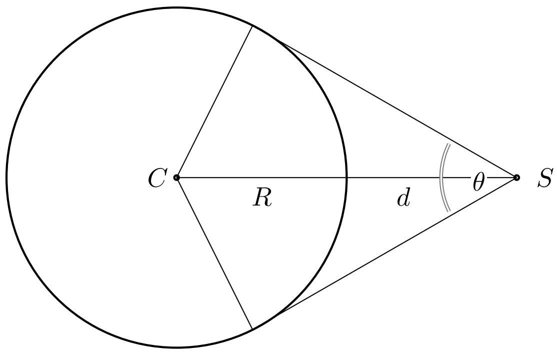
12) Find all real solutions to the equation

$$9^x - 3^x - 8 = 0.$$

12) ________

Express your answer as a single logarithm with base 3.

13) A satellite is orbiting a small, spherical planetoid of radius R at a distance d above the surface. If θ is the angle formed by the tangent lines from the satellite to the planetoid, what is $\sin\theta$ when $d = R$?



13) ________

14) On what interval is the following function increasing?

$$f(x) = \log_{1/2}\left(x^2 - 5x + 6\right)$$

14) ________

(a) $(5/2, \infty)$ (b) $(3, \infty)$
(c) $(-\infty, 5/2)$ (d) $(-\infty, 2)$

15) Find $\sin(2x)$ if $\sin(x) = 17\cos(x)$. **15)** ________

16) How many complex solutions to $z^{10} = i$ lie in the first quadrant of the Complex Plane? **16)** ________

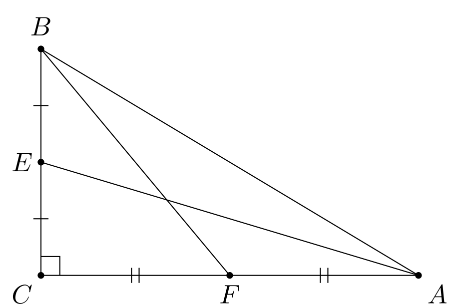
17) What is the minimum value of the function

17) ________

$$f(x) = \sqrt{x^2 - 1} + 2^{\sqrt{x^2+5x+4}}?$$

18) In right triangle $\triangle ABC$, AE is the median from $\angle A$ to its opposite leg, and BF is the median from $\angle B$ to its opposite leg. Find the numerical value of

$$\frac{m(AE)^2 + m(BF)^2}{m(AB)^2}.$$



18) ________

19) You have a deck with N cards numbered consecutively from 1 to N. If you shuffle the deck and draw 3 cards at random (without replacement), what is the probability that the third card drawn lies between the first two? **19)** ________

20) Let a and b be the real roots of the equation

$$x^2 - 3^{2011}x + 3^{4020} = 0.$$

20) ________

Find the exact value of

$$\log_3\left(\frac{a^3 + b^3}{2}\right).$$