Luzerne County Mathematics Contest

Luzerne County Council of Teachers of Mathematics
Wilkes University – 2018 Junior Examination
(Section 1)

NAME:_____   ADDRESS:_____

SCHOOL:_____   CITY/ZIP:_____

TELEPHONE:_____

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) What is the domain of the function $f(x) = \log\left(\dfrac{x-1}{2x+1}\right) - \sqrt{x+1}$? Express your answer in interval notation.

1) _____

2) Consider the following square. Find a formula for the length of the diagonal, d, as a function of the area of the square, A.
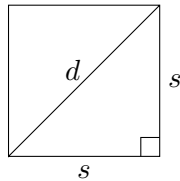
2) $d = $ _____

3) If $x = (1-\pi)^0 - \left|-\frac{1}{2}\right|$, then what is $\left(x - 2 - \dfrac{5}{x+2}\right)\left(\dfrac{2x+4}{x^2-3x}\right)$ equal to?

3) _____

4) Write the complex number $\dfrac{2+i}{3+2i}$ in the standard form $a+bi$.

4) _____

5) Find all real solutions to

$$x^{\frac{2}{3}} - 7x^{\frac{1}{3}} + 10 = 0.$$

5) $x = $ _____

6) A circle has center $(1, y_0)$ and passes through the points $(3,2)$ and $(5,4)$. Find y_0.
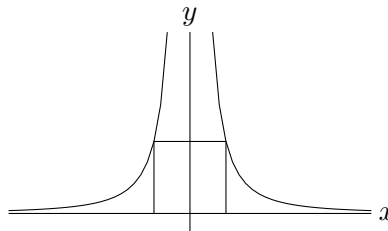
6) $y_0 = $ _____

7) If

$$f(x) = \begin{cases} x^2 - 2, & x \le 0 \\ 2x - 6 + \ln(x), & x > 0 \end{cases},$$

how many zeros does f have?

7) _____

8) Find and simplify $\dfrac{f(x+h) - f(x)}{h}$ $(h \ne 0)$ for $f(x) = \dfrac{1}{x^2}$.

8) _____

9) How many *distinct* rearrangements of the letters in the word "llamas" can be made?

9) _____

10) A square has its base on the x-axis, and one vertex on each branch of the curve $y = 1/x^2$. What is its area?
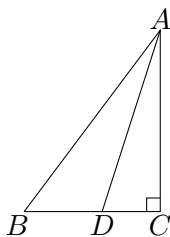

10) _____

(OVER)

11) Evaluate $\sin \dfrac{\pi}{12}$.

11) _____

12) In right triangle $\triangle ABC$, $m(\overline{AB}) = 5$ and $\overline{AD}$ bisects $\angle BAC$. If $m(\overline{CD}) = \sqrt{3}$, then what is the area of $\triangle ABD$?

12) _____

13) The set of complex numbers z that satisfy

$$|z + 3i| = 3|z|$$

13) $c =$ _____, $r =$ _____

is a circle in the complex plane. Find the complex number, c, that is its center along with its radius, r.

14) Find all solutions to
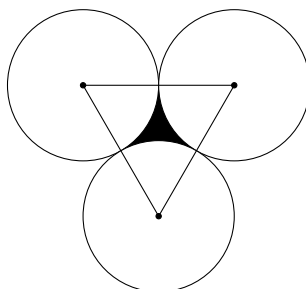
$$\sin^2 x - \cos^2 x = 0$$

14) $x =$ _____

in the interval $[0, 2\pi]$.

15) What is the area between the three circles (each of radius 1) shown below?

15) _____

16) Find all real solutions to

16) $x =$ _____

$$4^{\log_2(\sqrt{2}\, x)} + e^{4\ln(x)} - 3 = 0.$$

17) Find the value of $a^3 + \frac{1}{a^3}$ if $(a + \frac{1}{a})^2 = 3$.

17) _____

18) Find all solutions to

$$2\sin^2 \theta + 3\sin \theta - 2 = 0$$

18) $\theta =$ _____

that lie in the range $0 \le \theta < 2\pi$.

19) The expression

$$\sqrt{2\sqrt{2\sqrt{2\sqrt{\cdots}}}}$$

19) _____

simplifies to what integer?

20) How many real numbers satisfy the equation

20) _____

$$(x^2 + 2x)^{x^2 - 3x + 2} = 1?$$

Luzerne County Mathematics Contest
Luzerne County Council of Teachers of Mathematics
Wilkes University – 2018 Junior Examination
(Section 2)

NAME:_____   ADDRESS:_____

SCHOOL:_____   CITY/ZIP:_____

TELEPHONE:_____

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) Find the formula for the inverse function of

$$f(x) = \frac{2x - 5}{x + 4}.$$

1) $f^{-1}(x) =$ _____

2) Simplify the following expression.

$$(3 - \pi)^0 - \left(\frac{1}{3}\right)^{-1} + \left|2 - \sqrt{8}\right| + 2\cos 45°$$

2) _____

3) A square has diagonal of length 4. If a circle has the same circumference as the perimeter of the square, what is its radius?

3) $r =$ _____

4) If $x^2 + kx + 16$ (with $k > 0$) is a perfect square, then k must be equal to

(a) 16   (b) 4
(c) 8   (d) 32

4) _____

5) Find all real solutions to

$$\sqrt{x} + \sqrt[4]{x} - 6 = 0.$$

5) $x =$ _____

6) Which of the following is a square root of $-i$?

(a) i (b) $-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$
(c) $\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ (d) $-i$ has no square roots

6) _____

7) What is the maximum value attained by $f(x) = \frac{3}{5}\sin x - \frac{4}{5}\cos x$?

7) _____

8) If

$$f(x) = \begin{cases} \sqrt{x}, & x > 0 \\ x^2, & x \leq 0 \end{cases},$$

what is the value of $(f \circ f)(-2)$?

8) _____

9) The natural number 337500 factors as

$$337500 = 2^2 3^3 5^5.$$

9) _____

How many distinct positive divisors does 337500 have other than 1 and itself?

10) Consider the following recursively defined function on the integers.

$$f(n) = \begin{cases} f(n-1) + 2f(n-2), & n \neq 0 \text{ or } 1 \\ 1, & n = 0 \text{ or } 1 \end{cases}$$
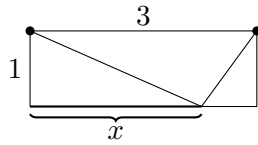
10) _____

Compute $f(6)$.

(OVER)

11) Find all real solutions to

$$\ln(x^2 - 1) - \ln(x + 1) - \ln(4) = 0.$$

11) $x =$ _____

12) You are at the back of a rectangular theater with the dimensions shown below. How far should you stand from the left wall so that the sound from the left front speaker takes twice as long as the sound from the right front speaker to reach you? In the figure below, x represents the distance to the left wall.

12) $x =$ _____

13) Find all real solutions to the following system of equations.

$$\begin{cases} xy - y - 2x + 2 = 0 \\ y^2 - 9xy + 20x = 0 \end{cases}$$

13) _____

List your answers as ordered pairs.

14) Find all real solutions to

$$\log_{2x+3}(19 - 6x) = 2.$$

14) $x =$ _____

15) For what base, b, is the following equation true.

$$(11.\overline{3})_b = (14.25)_{10}$$

15) $b =$ _____

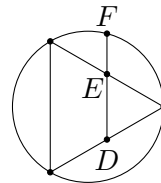**16)** If x and y satisfy

$$\begin{cases} x + 2y - 3 \le 0 \\ x + 3y - 3 \ge 0 \\ y \le 1 \end{cases},$$

then what is the largest value of $z = 2x + y$?

16) _____

17) An equilateral triangle is inscribed in a circle. Let D and E be midpoints of two of its sides, and let F be the point where the line from D through E meets the circle. What is the ratio DE/EF?

17) _____

18) What is the maximum value of the function

$$f(x) = \sqrt{3}\sin(2x) - 2\sin^2(x)?$$

18) _____

19) What is the coefficient of x^2 in the expression

$$(1 - x)^4(1 - \sqrt{x})^3?$$

19) _____

20) Suppose you take a random walk by the following process. Starting from position 0, you are allowed to take a single step left or right. To decide, you flip a fair coin. If the coin flip comes up heads, then you take a step to the right. Tails means you take a step to the left. You then repeat this process at your new position. After 10 steps, what is the probability that you are back at 0?

20) _____

Luzerne County Mathematics Contest

Luzerne County Council of Teachers of Mathematics
Wilkes University – 2018 Senior Examination
(Section 1)

NAME:_____   ADDRESS:_____

SCHOOL:_____   CITY/ZIP:_____

TELEPHONE:_____

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) Write the complex number $\dfrac{2+i}{-i}$ in the standard form $a+bi$.

1) _____

2) If -1 is a real solution to the equation $x^2 - 3x - a = 0$, then a is equal to

(a) 2   (b) -2
(c) 4   (d) -4

2) _____

3) Suppose θ is an angle in Quadrant II and

$$\sin\theta = \frac{1}{5}.$$

What is $\tan\theta$? Write your answer so that no radicals appear in the denominator.

3) _____

4) Find the real solutions to $2x^4 - 3x^2 - 2 = 0$.

4) $x = $ _____

5) What is the ratio of the area of a circle to the area of an inscribed square?

5) _____

6) What rational number is the solution to

$$8^{x+1} = 4^{5-x}?$$

6) $x = $ _____

7) Simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ $(h \neq 0)$ for the function $f(x) = x^2 + 2x$.

7) _____

8) Find all real solutions to

$$|x^2 - 2x| = 3x - 6.$$

8) $x = $ _____

9) Suppose you are rolling two 6-sided dice. What is the probability of getting a sum of 6?

9) _____

10) What rational number does the following expression reduce to?

$$\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$$
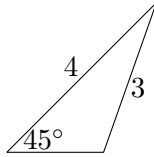
10) _____

(OVER)

11) Express $\log_a \dfrac{a^2 b^3}{\sqrt{c}}$ in terms of $\log_a(b)$ and $\log_a(c)$.

11) _____

12) Find the next element of the following sequence.

$$1, -\frac{3}{2}, \frac{5}{4}, -1, \frac{9}{11}, -\frac{11}{16}, \frac{13}{22}, \underline{\quad}$$

12) _____

13) A triangle has an angle of $45°$ with an adjacent side of length 4 and an opposite side of length 3.

13) _____

If the triangle has an obtuse angle, what is the length of the third side?

14) What integer does the following expression reduce to?

$$\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$$

14) _____

15) Find all real solutions to
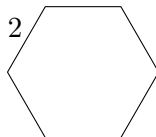
$$\log_2(x+2) + \log_2(x-2) = 5.$$

15) $x =$ _____

16) If

$$f(x) = 2\cos(x)\big(\sin(x) + \cos(x)\big),$$

then what is the maximum value of f?

16) _____

17) What is the area of a regular hexagon of side–length 2?

17) _____

18) If $x > 0$, $y > 0$, and $\dfrac{2}{x} + \dfrac{3}{y} = 1$, then the smallest value that the quantity

$$\frac{x}{2} + \frac{y}{3}$$

can achieve is which of the following?

18) _____

(a) 1   (b) 2
(c) 4   (d) 25/6

19) If the partial sums of a sequence $\{a_n\}_{n=1}^{\infty}$ are given by $S_n = n^3$, then a_4 must be equal to which of the following?

19) _____

(a) 37   (b) 27
(c) 64   (d) 91

20) Iterated exponents are defined by $a^{b^c} = a^{(b^c)}$. If x is a real number that satisfies

$$2^{2^x} + 4^{2^x} = 56,$$

20) _____

what is the value of $2^{2^{2^x}}$?

Luzerne County Mathematics Contest
Luzerne County Council of Teachers of Mathematics
Wilkes University – 2018 Senior Examination
(Section 2)

NAME:_____   ADDRESS:_____

SCHOOL:_____   CITY/ZIP:_____

TELEPHONE:_____

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) If set $P = \{x\mid 2 \leq x < 4\}$ and set $Q = \{x\mid x \geq 3\}$, then $P \cap Q$ is

    (a) $\{x\mid 3 \leq x < 4\}$ (b) $\{x\mid 3 < x < 4\}$
    (c) $\{x\mid 2 \leq x < 3\}$ (d) $\{x\mid 2 \leq x \leq 3\}$

1) _____

2) Convert the following point in polar coordinates to rectangular coordinates.

$$\left(7, \frac{3\pi}{4}\right)$$

2) _____

3) Find all real solutions to the following inequality. Express your answer in interval notation.

$$|3 - 4x| \leq 5$$

3) _____

4) Evaluate $2i^{100} + i^{202}$.

4) _____

5) A polynomial of the form $x^2 + bx + c$ with b and c real has $1 + \sqrt{2}i$ as a root. Find b and c.

5) $b =$ _____, $c =$ _____

6) Express the diameter of a circle, d, as a function of its area, A.

6) $d =$ _____

7) If a straight line passes through the center of the circle $x^2 + (y-3)^2 = 4$ and is perpendicular to $x + y + 1 = 0$, then which of the following is the equation of the line?

    (a) $x + y - 2 = 0$ (b) $x + y - 3 = 0$
    (c) $x - y + 2 = 0$ (d) $x - y + 3 = 0$

7) _____

8) Assuming $x \neq 0$, simplify the expression

$$\frac{\left(x^2\right)^4 x^8}{x^4 \left(x^3\right)^2}.$$

8) _____

9) There are five cards, two red and three black. You draw two cards, without replacement. What is the probability that both cards have the same color?

9) _____

10) Find all values of the real parameter m so that the equation

$$2x^2 - mx + m = 0$$

10) _____

has no real solutions. Express your answer in interval notation.

(OVER)

11) Solve the trigonometric equation

$$\sin^2 \theta = \cos^2 \theta$$

11) $\theta = $ _____

in the interval $0 \leq \theta \leq 2\pi$. List your answers in increasing order.

12) Find all real solutions to the following equation.

$$\log_3(x) + \frac{\ln(x+2)}{\ln(3)} = 1$$

12) $x = $ _____

13) Let $L = \max\{2^{35}, 5^{15}, 6^{14}\}$ and $S = \min\{2^{35}, 5^{15}, 6^{14}\}$. Find L and S.

13) $L = $ ____, $S = $ ____

14) What is the coefficient of $x^3 y^6$ in the expansion of

$$(x+y)(x-y)^8?$$

14) _____

15) Find the inverse function of $f(x) = \dfrac{x+b}{x-2}$, where $b > -2$ and constant.

15) $f^{-1}(x) = $ _____

16) Simplify the expression $\left(\sin \frac{\theta}{2} + \cos \frac{\theta}{2} \right)^2$. Your answer should only involve $\sin \theta$, $\cos \theta$, and/or $\tan \theta$.

16) _____

17) Find all real solutions to $3^{x^2+3} = 9^{2x}$.

17) $x = $ _____

18) Find all real solutions to

$$\sqrt{6-x} = 2x + 3.$$

18) $x = $ _____

19) In an arithmetic sequence $\{a_n\}_{n=1}^{\infty}$, $a_4 + a_8 = 16$. What is the partial sum of the first 11 terms?

(a) 58   (b) 88
(c) 176   (d) 143

19) _____

20) Find a formula for c in terms of a and b so that the linear system

$$\begin{cases} x + y + 5z = a \\ 2x + 3y + 13z = b \\ x + 2y + 8z = c \end{cases}$$

20) $c = $ _____

is guaranteed to have at least one solution.