Luzerne County Mathematics Contest

Luzerne County Council of Teachers of Mathematics
Wilkes University – 2019 Junior Examination
(Section 1)

NAME:_____ **ADDRESS:**_____

SCHOOL:_____ **CITY/ZIP:**_____

TELEPHONE:_____

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) Which of the following equals $4\sin(45°) + (\pi - 2)^0 - \sqrt{18} + \left|-1\right|$?

(a) $3 - \sqrt{2}$ (b) $3 + \sqrt{2}$
(c) $2 - \sqrt{2}$ (d) $2 + \sqrt{2}$

1) _____

2) Express the following complex number in standard form $a + bi$.

$$\frac{6 - 9i}{2i^5}$$

2) _____

3) How many ways can you choose 3 objects from a collection of 25 distinguishable objects (assuming the order they are selected is unimportant)?

3) _____

4) A gold bar is a rectangular solid measuring $2 \times 3 \times 4$. It is melted down, and three equal cubes are constructed from this gold. What is the length of a side of each cube?

4) _____

5) If $\dfrac{2}{x} = \dfrac{y}{3} = \dfrac{x}{y}$, then what is the value of y?

5) _____

6) The function

$$f(x) = \frac{x - 3a}{x - 2b}$$

has the property that $f(5) = 0$ and $f(3)$ is undefined. What is the value of $f\left(\frac{1}{3}\right)$?

6) _____

7) What is the smallest positive number b such that

$$2x^2 + bx + 5$$

has real roots?

7) _____

8) A quadratic function $f(x)$ satisfies $f(2) = 3$, $f(1) = 1$, and $f(x) \geq 1$ for all x. What is $f(3)$?

8) _____

9) Which of the following equals $\sin(\arctan(5))$?

(a) $\frac{5\sqrt{26}}{26}$ (b) $\frac{\sqrt{26}}{26}$
(c) $\frac{\sqrt{26}}{5}$ (d) $\sqrt{26}$

9) _____

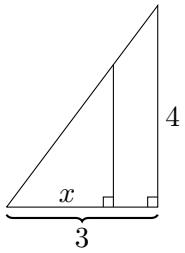
10) Find the value of a so that the graph $y = g(x)$ is an unbroken curve.

$$g(x) = \begin{cases} ax^2 + 8, & x < 3 \\ a, & x \geq 3 \end{cases}$$

10) _____

(OVER)

11) Determine the length x of the base of the smaller triangle in the diagram below so that the smaller triangle has half the area of the larger triangle.



11) _____

12) Find the circumference of a circle inside of which is inscribed a triangle with side lengths $3, 4$, and 5.

12) _____

13) How many of the 3-digit numbers from 100 to 999 have the property that all digits are perfect squares? For example, 100 is the smallest such number and 999 is the largest.

13) _____

14) A box of crayons comes with 10 crayons in a random assortment of 4 colors. What is the minimum number of boxes we can buy to be certain that we have 13 of one color?

14) _____

(a) 4 (b) 5
(c) 6 (d) 13

15) If integers a and b satisfy

$$\left(a^{\frac{1}{2}}b^{\frac{1}{3}}\right)^6 = 432,$$

what is the value of ab?

15) _____

(a) 8 (b) 16
(c) 12 (d) 24

16) If $x > y > 0$ and $2\ln(x-y) = \ln(2x) + \ln(y)$, what is the value of $\dfrac{x}{y}$?

16) _____

17) If

$$f(x) = \ln\left(\sqrt{1+x^2} - x\right) + 1$$

and $f(a) = 4$, what is $f(-a)$?

17) _____

18) What is the coefficient of x^2 in the expansion of

$$\left(x - \frac{1}{2\sqrt{x}}\right)^5 ?$$

18) _____

19) Find the angle α in Quadrant I satisfying

$$\cos\alpha = \frac{1}{2}\sqrt{2+\sqrt{3}}.$$

Give your answer in degrees.

19) _____

20) A cubical die with the numbers 1, 2, 3, 4, 5, and 6 on its faces is loaded in such a way that the probability that the number i turns up is proportional to i, $i = 1, 2, 3, 4, 5, 6$. What is the probability that an odd number turns up?

20) _____

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) Which of the following equals $\dfrac{\log_{10}(7)}{\log_{10}\left(\frac{1}{7}\right)}$?

(a) 1 (b) $\sqrt[10]{7}$
(c) -1 (d) $-\sqrt[10]{7}$

1) _____

2) Let n be the integer

$$n = \left(2^{100} - 1\right)\left(2^{100} + 1\right).$$

2) _____

What is the smallest positive integer k such that $n + k$ is a perfect square?

3) Leo is three times as old as Amy. In 7 years, he will be twice as old as she. How old is Leo?

3) _____

4) What is the distance between the points of intersection of the curves $y = x^2 + x$ and $y = 2x + 2$?

4) _____

5) What is the domain of the following function? Express your answer in interval notation.

$$f(x) = \ln(\ln(x^2 - 1))$$

5) _____

6) Find $\cos\theta$ if $\sin\theta = \dfrac{2}{7}$ and θ is in quadrant II.

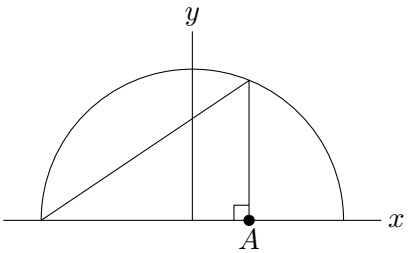
6) _____

7) Which of the following equals i^{403}?

(a) 1 (b) i
(c) -1 (d) $-i$

7) _____

8) A right triangle has two vertices on the semicircle of radius 2 centered at the origin as shown below. If the hypotenuse of the right triangle is 3 units, find the x–coordinate of the vertex labeled A.



8) _____

9) Find all solutions to $\sin^2 x = 3\cos^2 x$ in the interval $0 \le x \le \pi$.

9) _____

10) If $9^{-x} = 7$, what rational number is equal to 27^{2x+1}?

10) _____

(OVER)

11) How many ordered pairs of non–negative integers (x, y) satisfy $xy - x + y - 1 = 48$?

11) _____

12) Let

$$f(x) = \frac{x}{x+2}$$

for $x \neq -2$. Find all values x for which $f(f(x)) = x$.

12) _____

13) In $\triangle ABC$, $\overline{BC}$ has length 1, $\overline{AC}$ has length 5, and

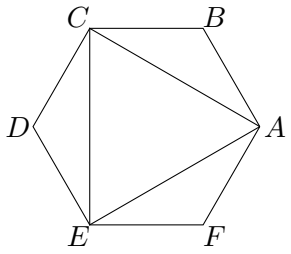
$$\cos\left(\frac{m\angle C}{2}\right) = \frac{\sqrt{5}}{5}.$$

Which of the following equals the length of $\overline{AB}$?

(a) $4\sqrt{2}$ (b) $\sqrt{30}$
(c) $\sqrt{29}$ (d) $2\sqrt{5}$

13) _____

14) A regular hexagon has sides of length 6. What is the area of $\triangle ACE$?



14) _____

15) How many distinct five–letter permutations can be made from the letters in the word

ADDITION ?

15) _____

16) If $\sin\alpha + \cos\beta = 1$ and $\cos\alpha + \sin\beta = 0$, then what is the value of $\sin(\alpha+\beta)$?

16) _____

17) For what integer b does the number $\frac{1}{4}$ have the representation $0.\overline{2}$ in base b?

(a) 10 (b) 9
(c) 8 (d) 7

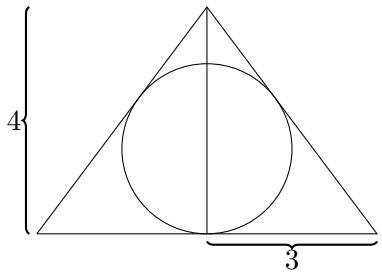
17) _____

18) What is the value of the following continued fraction

$$3 + \frac{1}{4 + \frac{1}{3 + \frac{1}{4 + \cdots}}}?$$

18) _____

19) What is the radius of the largest sphere that will fit inside a right circular cone with height 4 and a base radius of 3?



19) _____

20) What is the probability that three random real numbers chosen independently and uniformly from the interval $0 \leq x \leq 1$ have a sum less than 1?

20) _____

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) Fully simply the following expression. Your final answer should only use positive exponents.

$$\left(\frac{24a^3b^{-8}}{6a^{-5}b}\right)^{-2}.$$

1) _____

2) Write the following complex number in the standard form $a + bi$.

$$\frac{5-i}{3+2i}$$

2) _____

3) The perimeter of a rectangle is 20. A second rectangle is three times as long as the first, and twice as wide. The perimeter of the second rectangle is 50. Find the area of the second rectangle.

3) _____

4) What integer is equal to $\log_3(\log_3(\log_3(27)))$?

(a) -1 (b) 0
(c) 1 (d) 2

4) _____

5) If angle θ is in Quadrant IV and $\cos\theta = \frac{3}{7}$, what is $\tan\theta$?

5) _____

6) If $f(x) = x + 2$, $g(x) = x^2 - 3$, and $h(x) = x - 5$, what is the value of $f(g(h(10)))$?

6) _____

7) Where is the polynomial

$$p(x) = -x^3 + x^2 + 6x$$

7) _____

strictly positive? Give your answer in interval notation.

8) Which of the following equals $\dfrac{2^{2019} - 2^{2015}}{2^{2016} + 2^{2020}}$?

(a) 31/66 (b) 7/18
(c) 8/17 (d) 15/34

8) _____

9) What rational number does the following expression reduce to?

$$\cos\left(\operatorname{arccot}\left(\frac{3}{4}\right)\right)$$

9) _____

10) What is the domain of the following function? Write your answer in interval notation.

$$f(t) = \frac{\sqrt{6+t-t^2}}{t-1} - \log_{10}(t+2)$$

10) _____

(OVER)

11) In an arithmetic sequence $\{a_n\}$, $a_1 = 3$ and $a_3 + a_5 = 36$. What is a_{10}? **11)** _____

12) Find a formula for C in terms of A and B so that the system below has an infinite number of solutions. **12)** _____

$$\begin{cases} x + 2y + 3z = A \\ 2x - 4y + z = B \\ 4x + 0y + 7z = C \end{cases}$$

13) A line containing the points $(2, 1)$ and $(-8, a)$ is parallel to the line containing the points $(7, a + 1)$ and $(8, 0)$. Find the value of a. **13)** _____

14) Which of the following is the period of the function **14)** _____

$$f(x) = \frac{\tan(x)}{1 + \tan^2(x)}?$$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) π (d) 2π

15) Let $f(x) = \dfrac{x}{x+3}$ for $x \neq -3$. List all values of x for which $f\big(f(x)\big) = x$. **15)** _____

16) How many distinct arrangements can be made using all of the following letters? **16)** _____

ANAGRAM

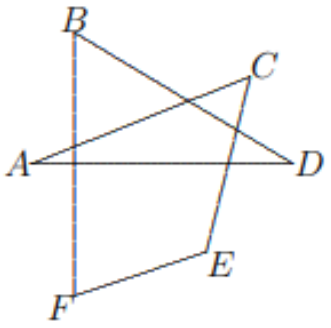
17) A standard deck of 52 cards has four suits each with 13 cards of different face values. How many distinct 5 card hands have four cards with the same face value (i.e. a hand with 4–of–a–kind)? **17)** _____

18) What rational number is equal to **18)** _____

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{99 \cdot 100}?$$

19) If $x = 2 + \sqrt{3}$, find an integer equal to $x^4 + \dfrac{1}{x^4}$. **19)** _____

20) In the diagram below, angle D equals 30 degrees. What is the sum of angles A, B, C, D, E, and F in degrees? **20)** _____



(a) 360° (b) 450°
(c) 540° (d) 630°

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) If $A = \{x \mid |x| < 2\}$ and $B = \{-2, 0, 1, 2\}$, then what is $A \cap B$?

(a) $\{0, 1\}$ (b) $\{-1, 0, 1\}$
(c) $\{-2, 0, 1, 2\}$ (d) $\{-1, 0, 1, 2\}$

1) _____

2) If $5x + 6y = 2019$ and $6x + 5y = 9102$, then what is the value of $x + y$? **2)** _____

3) If $a - b = 2\sqrt{3}$, then which of the following is equal to $\left(\dfrac{a^2 + b^2}{2a} - b\right)\dfrac{a}{a - b}$?

(a) $\sqrt{3}$ (b) $2\sqrt{3}$
(c) $3\sqrt{3}$ (d) $4\sqrt{3}$

3) _____

4) The hyperbolic cosine is defined as $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$. What rational number is equal to $\cosh(\ln(5))$? **4)** _____

5) Compute $|z|$ for the complex number $z = \dfrac{1 - i}{1 + i} + 2i$. **5)** _____

6) Let $x = \sqrt{2}$. What integer is equal to the following expression? **6)** _____

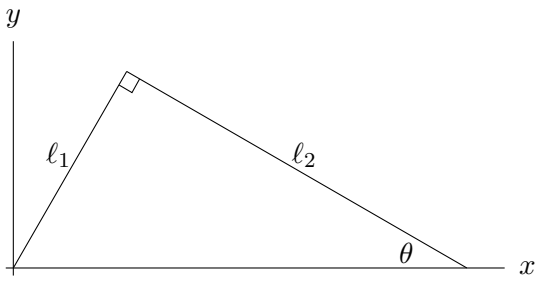
$$-\ln(\sin(x)) + \ln(\cos(x)) + \ln(\tan(x))$$

7) If $a = 2b + 2$, what integer equals $\dfrac{3^a}{9^b}$? **7)** _____

8) A circle has center $(x_0, 6)$ and passes through the points $(0, 1)$ and $(5, 6)$. Find x_0. **8)** _____

9) What is the domain of $f(x) = \sqrt{7 + 13x - 2x^2}$? Express your answer in interval notation. **9)** _____

10) Line ℓ_1 with slope $m = \sqrt{3}$ intersects line ℓ_2 at a right angle in Quadrant I. At what angle θ does ℓ_2 intersect the x–axis? Express your answer in radians.



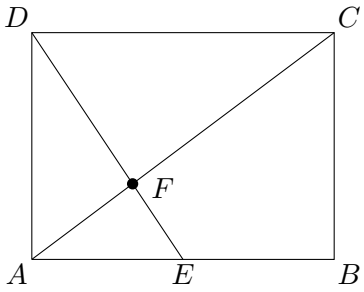
10) _____

(OVER)

11) How many real solutions does $|2x^2 - x - 1| = x$ have? **11)** _____

12) For what value of n is it true that $3^1 \cdot 3^2 \cdot 3^3 \cdots 3^n = 3^{91}$? **12)** _____

13) In rectangle $ABCD$, E is the midpoint of $\overline{AB}$. If the length of $\overline{AB}$ is 4 and the length of $\overline{AD}$ is 3, then what is the length of $\overline{CF}$?



13) _____

14) How many of the 2-digit numbers from 10 to 99 have the property that both digits are perfect squares? For example, 10 is the smallest such number and 99 is the largest. **14)** _____

15) Suppose that $f(x)$ is an odd function with the property $f(1-x) = f(1+x)$. If $f(1) = 2$, then what is the value of $f(1) + f(2) + f(3) + \cdots + f(50)$?

(a) -50 (b) 0
(c) 2 (d) 50

15) _____

16) If $\sin x + \cos x = \dfrac{1}{3}$, what is the value of $\sin^3 x + \cos^3 x$? **16)** _____

17) Let $S_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n-1}n$. What is the value of $\displaystyle\sum_{n=1}^{2019} S_n$? **17)** _____

18) Mr. Thomas wants to divide the seven students in his algebra class into three teams of at least two students each. How many different ways are there for him to divide his students into teams? **18)** _____

19) If $f(x) = \cos(x) - \sin(x)$ is decreasing on $[-a, a]$, what is the largest possible value of a?

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{4}$ (d) π

19) _____

20) We have an urn containing 4 red balls, 6 yellow balls, and 3 green balls. If we remove 5 balls from the urn (without replacement), what is the probability of getting exactly 2 yellow balls? **20)** _____

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) 1) C

2) 2) $-\frac{9}{2}-3i$

3) 3) 2300

4) 4) 2

5) 5) $\sqrt[3]{18}$

6) 6) 17/4

7) 7) $2\sqrt{10}$

8) 8) 9

9) 9) a

10) 10) −1

(OVER)

11) $\frac{3\sqrt{2}}{2}$

12) 5π

13) 48

14) b

15) c

16) $2+\sqrt{3}$

17) -2

18) $5/2$

19) 15°

20) $3/7$

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(Section 2)

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) 1) C

2) 2) 1

3) 3) 21

4) 4) $3\sqrt{5}$

5) 5) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, +\infty)$

6) 6) $-\frac{3\sqrt{5}}{7}$

7) 7) d

8) 8) 1/4

9) 9) $\frac{\pi}{3}, \frac{2\pi}{3}$

10) 10) $27/343$

(OVER)

11) 11) 10

12) 12) $-1, 0$

13) 13) a

14) 14) $27\sqrt{3}$

15) 15) 2040

16) 16) $-1/2$

17) 17) b

18) 18) $\frac{3+2\sqrt{3}}{2}$

19) 19) $3/2$

20) 20) $1/6$

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1) 1) $\frac{b^{18}}{16a^{16}}$

2) 2) $1-i$

3) 3) 150

4) 4) b

5) 5) $-\frac{2\sqrt{10}}{3}$

6) 6) 24

7) 7) $(-\infty,-2)\cup(0,3)$

8) 8) d

9) 9) $3/5$

10) 10) $(-2,1)\cup(1,3]$

(OVER)

11) 11) 48

12) 12) 2A + B

13) 13) -11/9

14) 14) C

15) 15) -2, 0

16) 16) 840

17) 17) 624

18) 18) 99/100

19) 19) 194

20) 20) a

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) 1) a

2) 2) 1011

3) 3) a

4) 4) 13/5

5) 5) 1

6) 6) 0

7) 7) 9

8) 8) 0

9) 9) $[-\frac{1}{2}, 7]$

10) 10) $\pi/6$

(OVER)

11) 11) 2

12) 12) 13

13) 13) $10/3$

14) 14) 12

15) 15) c

16) 16) $13/27$

17) 17) 1010

18) 18) 105

19) 19) a

20) 20) $175/429$