LUZERNE COUNTY MATHEMATICS CONTEST

Luzerne County Council of Teachers of Mathematics
Wilkes University - 2009 Junior Examination
(Section I)

NAME: _____ **Address:** _____

SCHOOL: _____ **City/ZIP:** _____

Telephone: _____

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) Suppose that a square has sides of length s units. If each side is increased by 1.5 units, express the increase, I, in the area of the square in terms of s. **1)** $I =$ _____

2) What is the area of the triangle with vertices (-4, -2), (-4, 8), and (6, -2)? **2)** _____

3) What is the sum of all the prime numbers between 1 and 37 inclusive? **3)** _____

4) The harmonic mean of two numbers is the reciprocal of the average of the reciprocals of the two numbers. Find the harmonic mean of 7 and 9. **4)** _____

5) There are 14 juniors and 23 seniors in the service club. The club is to send 4 representatives to the state conference. If the members of the club decide to send 2 juniors and 2 seniors, then how many groupings are possible? **5)** _____

6) If $x \log_3 2 = 1$, then $2^x =$ _____ . **6)** _____

7) Find all real numbers x such that the distance between $(3, 4)$ and $(x, 2)$ is equal to 5 units. **7)** _____

8) If the straight line $x + my + 1 = 0$ is perpendicular to the straight line $m^2x - 2y - 1 = 0$, then m is **8)** _____

(a) $\sqrt[3]{2}$ **(b)** 0 or 2 **(c)** 2 **(d)** 0 or $\sqrt[3]{2}$

9) Express the complex number $\left(\frac{2i}{1+i}\right)^2$ in the form $a + bi$ **9)** _____

where a and b are real numbers.

10) Re-express $\frac{10}{\sqrt[3]{5}}$ by rationalizing the denominator. **10)** _____

(OVER)

11) If $f(x) = x^2 - 2ax + 2$ and $f(x) \geq a$ when $x \in [-1, \infty)$, then a satisfies which of the following? **11)**_____

(a) $-1 < a < 1$ **(b)** $-2 \leq a \leq 1$

(c) $-3 \leq a \leq -2$ **(d)** $-3 \leq a \leq 1$

12) Find all real values x such that $9^x - 6 \cdot 3^x - 7 = 0$. **12)** $x =$_____

13) Albert had a bag of apples. He gave $\frac{1}{4}$ of the apples to Sara. **13)**_____

He then gave $\frac{1}{3}$ of what was left to Gerald. After giving $\frac{1}{2}$ of the remaining apples to his sister and eating one apple, Albert had 12 apples left. How many apples were originally in the bag?

14) What is the range of $f(x) = \frac{2}{14 + e^x} + 3$? Express your answer in interval notation. **14)**_____

15) If $\sin\theta + \cos\theta = \frac{1}{5}$, then $\sin 2\theta =$ _____. **15)**_____

16) The constant term in the expression $\left(\sqrt{x} - \frac{1}{x}\right)^9$ is **16)**_____

(a) -36 **(b)** 36 **(c)** -84 **(d)** 84

17) If $a_n = \frac{1}{(n+1)(n+2)}$, then the partial sum s_n of the sequence $\{a_n\}_{n \geq 1}$ is **17)**_____

(a) $\frac{1}{2}$ **(b)** $\frac{1}{2} - \frac{1}{n+2}$ **(c)** $\frac{1}{2n+3}$ **(d)** $\frac{2n+3}{(n+1)(n+2)}$

18) Let $A = \{x \mid 0 < x < 9,\ x \text{ is a prime}\}$ and $B = \{x \mid 0 < x < 9,\ x \in \mathbb{N}\}$, then the number of sets S satisfying $A \subset S \subseteq B$ is _____. **18)**_____

19) If $a = \frac{1}{2 - \sqrt{5}}$, $b = \frac{1}{2 + \sqrt{5}}$ then $a + b + ab$ is equal to **19)**_____

(a) $1 + 2\sqrt{5}$ **(b)** $1 - 2\sqrt{5}$ **(c)** -5 **(d)** 3

20) Find the domain of $\sqrt{\frac{x^2 + 4x + 3}{x - 5}}$. **20)**_____

LUZERNE COUNTY MATHEMATICS CONTEST

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) Find the area of the circle that passes through (6, 4) and has as its center (2, 3). **1)**_____

2) Express $i^{27} + 5i + (7 + 2i)(5 - 2i)$ as a real number. **2)**_____

3) If the length of the side of a square increases by 50%, then the area of the square increases by **3)**_____
(a) 100% **(b)** 50% **(c)** 300% **(d)** 125%

4) Find all real values x such that $3^{\frac{6}{\log_7 x}} = \frac{1}{27}$. **4)** $x =$_____

5) Three digits are chosen from 1, 2, 3, 4, 5 at one time. What is the probability that two odd digits remain? **5)**_____

6) If $f(x)$ satisfies $f(x + 2) = \frac{13}{f(x)}$ and $f(1) = 2$, then $f(99)$ is equal to _____ . **6)**_____

7) How many 3-element subsets does the set $\{1, 2, 3, 4, 5, 6, 7\}$ have? **7)**_____

8) Find all solutions to $\tan^4 2x - 9 = 0$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. **8)**_____

9) Find all real solutions to the equation $\frac{x^2 - 2}{x} + \frac{2x}{x^2 - 2} = 3$. **9)** $x =$_____

10) If $x^2 + mx - 15 = (x + 3)(x + n)$, then $n =$ **10)**_____
(a) -5 **(b)** 5 **(c)** -2 **(d)** 2

(OVER)

11) Find all real values of k such that $x = -4$ is a root of $P(x) = kx^2 + kx + 3$. **11)** $k =$ _____

12) The equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$ represents **12)** _____

(a) a circle with radius 5.

(b) an ellipse with eccentricity $\frac{4}{5}$.

(c) a circle with radius 3.

(d) an ellipse with eccentricity $\frac{3}{5}$.

13) Convert $(\sqrt{2}, -\frac{\pi}{4})$ from polar to rectangular coordinates. **13)** _____

14) Find the domain of $y = \sqrt{\log_{\frac{2}{5}} x - 1}$. **14)** _____

15) Find all real solutions to the equation $\sqrt{x} - 5\sqrt{\sqrt{x}} - 14 = 0$. **15)** $x =$ _____

16) Solve for x: $\frac{3}{4} \le \frac{1}{x - 3} < 5$. **16)** _____

17) A regular polygon having interior angles whose sum is 720° is called **17)** _____

(a) a quadrilateral. **(b)** an octagon. **(c)** a hexagon.
(d) a heptagon. **(e)** none of the above.

18) If $f(x) = \begin{cases} 2e^{x-1}, & x < 2 \\ \log_3(x^2 - 1), & x \ge 2 \end{cases}$, then $f\left(f(2)\right) =$ _____. **18)** _____

19) Let $f(x) = ax^3 + bx + 7$. If $f(5) = 3$, then what is $f(-5)$? **19)** _____

20) If m and n satisfy $m + 4\sqrt{mn} - 2\sqrt{m} - 4\sqrt{n} + 4n = 3$, **20)** _____

then $\frac{\sqrt{m} + 2\sqrt{n} - 8}{\sqrt{m} + 2\sqrt{n} + 2002}$ is equal to _____.

LUZERNE COUNTY MATHEMATICS CONTEST

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) What is the largest integer less than 2000 that is divisible by 7? **1)**_____

2) Suppose $A = \sqrt{BC + BD}$. Solve for B in terms of A, C, and D. **2)**_____

3) A square and a circle have equal areas. Express the radius of the circle, r, in terms of the side of the square, s. **3)** $r =$_____

4) The sum of three consecutive odd numbers is 15. What is the product of these numbers? **4)**_____

5) Compute $\lim_{x \to -4} \dfrac{5x + 20}{x^2 - 16}$. **5)**_____

6) Find all real solutions to $e^x - 8e^{-x} + 2 = 0$. **6)**_____

7) Let $f(x) = \dfrac{1 + x}{1 - x}$, $f_1(x) = f(x)$, and $f_{n+1}(x) = f\left(f_n(x)\right)$, then **7)**_____

$f_{2006}(x) =$

(a) x **(b)** $-\dfrac{1}{x}$ **(c)** $\dfrac{1 + x}{1 - x}$ **(d)** $\dfrac{x - 1}{x + 1}$

8) Find all real numbers x such that $x^2 < 2x + 3$ and $\log_2(x - 1) < 1$. **8)**_____

9) Express $0.1\overline{245}$ as a common fraction in lowest terms. **9)**_____

10) A rectangular sheet of metal is 20 inches wide. The length of a diagonal between opposite corners is 5 inches more than the length of the whole sheet. What is the length of the sheet of metal? **10)**_____ inches

(OVER)

11) How many ways are there to put 14 indistinguishable balls into 3 distinguishable urns if each urn is to contain at least one ball? **11)**_____

12) The number of real solutions to the system of equations **12)**_____

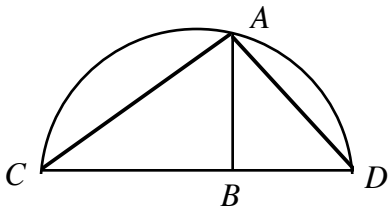
$$\begin{cases} |x| + y = 12 \\ x + |y| = 6 \end{cases}$$ is

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4

13) The remainder of $\left(x^{100} + 75x - 75B\right) \div \left(x - B\right)$ written in terms of B = _____ . **13)**_____

14) $f(x)$ satisfies $f(x+2) = \dfrac{1}{f(x)}$. If $f(1) = -5$, then what is $f\left(-f(5)\right)$? **14)**_____

15) $\overline{CD}$ is the diameter of a semicircle. A is a point on the semicircle, and $\overline{AB}$ is perpendicular to $\overline{CD}$. If $\overline{AB} = 20$ and $\overline{BD} = 10$, then $\overline{BC}$ = **15)**_____



(a) 45 **(b)** 30 **(c)** 15 **(d)** 40

16) Three line segments are randomly selected from five line segments with lengths 1, 3, 5, 7, and 9 (without replacement). What is the probability of forming a triangle using the three chosen segments? **16)**_____

17) If $\sin\alpha - \cos\alpha = -\dfrac{\sqrt{5}}{2}$, then $\tan\alpha + \dfrac{1}{\tan\alpha}$ = _____ . **17)**_____

18) Find the minimum value attained by y if $y = \sin^2 x - 3\sin x + 1$. **18)**_____

19) If $a \geq 0, b \geq 0,$ and $a + b = 4$, then **19)**_____

(a) $\dfrac{1}{ab} \geq \dfrac{1}{2}$ **(b)** $\dfrac{1}{a} + \dfrac{1}{b} \geq 1$

(c) $\sqrt{ab} \geq 2$ **(d)** $\dfrac{1}{a^2 + b^2} \geq \dfrac{1}{4}$

20) If $x^2 + y^2 = 1$, then what is the largest value of $3x + 4y$? **20)**_____

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

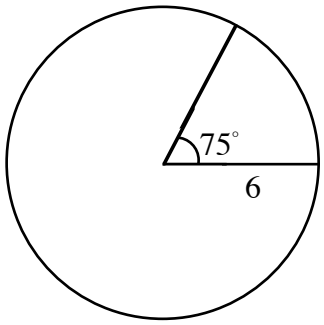
1) What is the greatest common divisor of 34,650 and 2,574? **1)**_____

2) What is the area of a square that has a diagonal length of $8\sqrt{2}$ units? **2)**_____

3) Find the equation of the line, in slope-intercept form, that is the perpendicular bisector of the line segment with endpoints (3, 7) and (-1, 5). **3)**_____

4) A rectangle is twice as long as it is wide. The area of the rectangle is 72 square units. What is the perimeter of the rectangle? **4)**_____

5) Find the area of the sector shown at the right. **5)**_____



6) Find the domain of the function $\sqrt{\log_{0.5}(3x-2)}$. **6)**_____

7) What is the maximum value of the function $f(x) = 3 - 8x - x^2$? **7)**_____

8) The inverse function of $f(x) = \sqrt{1 + 4x - x^2}$, $x \le 2$ is **8)**_____

(a) $f(x) = 2 + \sqrt{5 - x^2}$, $0 \le x \le \sqrt{5}$

(b) $f(x) = 2 + \sqrt{5 - x^2}$, $-\sqrt{5} \le x \le 0$

(c) $f(x) = 2 - \sqrt{5 - x^2}$, $0 \le x \le \sqrt{5}$

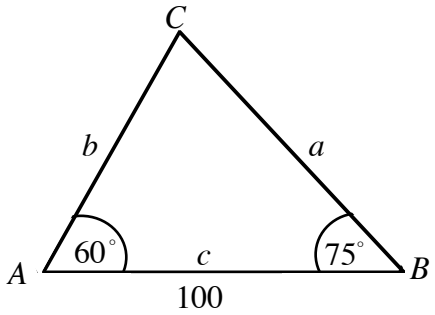
(d) $f(x) = 2 - \sqrt{5 - x^2}$, $-\sqrt{5} \le x \le 0$

9) Find all real values of x such that $5^x = 125^{4x+1}$. **9)**_____

10) What is the sum of the integers 57 through 88 inclusive? **10)**_____

(OVER)

11) Find the length of side a in the triangle below. **11)**_____



12) Bob has \$1.55 in change consisting of quarters, dimes, and nickels. If he has 2 more dimes than quarters, and twice as many nickels as quarters, how many nickels does Bob have? **12)**_____

13) If the equation $(m-2)x^2 - 2x + 1 = 0$ has a real solution, then **13)**_____

(a) $m < 3$ **(b)** $m \le 3$

(c) $m < 3$ and $m \ne 2$ **(d)** $m \le 3$ and $m \ne 2$

14) Find all real values x such that $e^{2x} - 5e^x - 24 = 0$. **14)** $x =$_____

15) If $f(x) = 1 - 2x$, $g[f(x)] = \dfrac{1-x^2}{x^2}$, $x \ne 0$, what is $g(\dfrac{1}{2})$? **15)**_____

16) If the sum of all the coefficients in the expansion of $\left(x + \dfrac{1}{x}\right)^n$ is 64, then what is the value of the constant term? **16)**_____

17) If the function $f(x) = \log_a\left(\dfrac{1-mx}{x-1}\right)$, $(a > 0,\ a \ne 1,\ m \ne 1)$ is an odd function, then what is the value of m ? **17)**_____

18) If a and b are real numbers, and $a^2 + b^2 = 2$, then **18)**_____

(a) $a + b < 2$ **(b)** $a + b \le 2$

(c) $a + b > 2$ **(d)** $a + b \ge 2$

19) If $\begin{cases} 2x + 3y \le 6 \\ x - y \ge 0 \\ y \ge 0 \end{cases}$, then what is the largest value of $3x + y$? **19)**_____

20) The even function $f(x)$ is a monotone function on $[0, a]$, $a > 0$. If $f(0) \cdot f(a) < 0$, then what is the number of roots of $f(x)$ on $[-a, a]$? **20)**_____