

# LUZERNE COUNTY MATHEMATICS CONTEST

Luzerne County Council of Teachers of Mathematics

Wilkes University - 2010 Junior Examination

(Section I)

NAME: \_\_\_\_\_

Address: \_\_\_\_\_

SCHOOL: \_\_\_\_\_

City/ZIP: \_\_\_\_\_

Telephone: \_\_\_\_\_

**Directions:** For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

- 1) Express  $\frac{(0.01)^3}{(0.5)^2}$  as a fraction in lowest terms. 1)  $\frac{1}{250,000}$
- 2) The average of three numbers is 21. If two of the numbers are 4 and 18, what is the third number? 2)  $41$
- 3) Reduce  $\frac{x^3 + 27}{x^2 - 2x - 15} \cdot \frac{2x - 10}{x^2 - 3x + 9}$  to lowest terms. 3)  $2$
- 4) If the perimeter of a square is  $\frac{1}{3}$  the area of the same square, then what is the length of the side of the square? 4)  $12$
- 5) Find the vertex of the parabola  $y = 2x^2 + 6x + 3$ . 5)  $\left(-\frac{3}{2}, -\frac{3}{2}\right)$
- 6) The sum of the real solutions to  $x^2 - 2|x| - 15 = 0$  is equal to:  
(a) 0 (b) -2 (c) 2 (d) 8 6)  $A$
- 7) Find all real solutions to  $\ln(x + 1) - \ln x - 5 = 0$ . 7)  $\frac{1}{e^5 - 1}$
- 8) Matt, Valerie, and Roy have a total of \$5.50 in nickels. Matt has 3 times as many nickels as Valerie and 2 times as many nickels as Roy. How many nickels does Roy have? 8)  $30$
- 9) The number of zeros of  $f(x) = \left(\frac{1}{2}\right)^x - \sin x$  on the interval  $[0, 2\pi]$  is  
(a) 1 (b) 2 (c) 3 (d) 4 9)  $B$
- 10) If  $|x + 2| + |x + 1| > k$  for all real numbers  $x$ , then what is the range of values for  $k$ ? 10)  $k < 1$

(OVER)

11) If  $\log_a 10 = B^x$  and  $B^x = 2$ , then  $a$  equals ?

11)  $a = \sqrt{10}$  \_\_\_\_\_

12) Suppose  $f(x) = x^2 + bx + c$  has exactly one real  $x$ -intercept and this  $x$ -intercept is also the  $x$ -intercept of the line  $y = 3x + 4$ , what is the value of  $b$  ?

12)  $b = \frac{8}{3}$  \_\_\_\_\_

13) There is a square  $\square ABCD$ .  $M$  and  $N$  are midpoints of  $\overline{BC}$  and  $\overline{CD}$ , respectively. A point is selected at random inside the square. What is the probability that this point lies within  $\triangle MCN$  ?

13)  $\frac{1}{8}$  \_\_\_\_\_

14) If  $f(x) = 4x^2 - kx - 8$  is a monotone function on  $[5, 8]$ , then what is the range of  $k$  ?

14)  $k \leq 40$  or  $k \geq 64$  \_\_\_\_\_

15) Suppose  $f(x) = 1 - \sqrt{x-1}$ , where  $x \geq 1$ , then  $f^{-1}(x)$  is

15)  $C$  \_\_\_\_\_

- (a)  $(x-1)^2 + 1, x \in \mathbb{R}$       (b)  $(x-1)^2 - 1, x \in \mathbb{R}$   
(c)  $(x-1)^2 + 1, x \leq 1$       (d)  $(x-1)^2 - 1, x \leq 1$

16) What is the largest perfect square divisor of 8,432 ?

16)  $16$  \_\_\_\_\_

17) If a complex number  $z$  satisfies  $|z+i| + |z-i| = 2$ , then the smallest value attained by  $|z+i+1|$  is :

17)  $B$  \_\_\_\_\_

- (a) 2      (b) 1      (c)  $\sqrt{5}$       (d) 3

18) What is the value of  $\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$  if  $\tan\theta = \sqrt{2}$  ?

18)  $-3 - 2\sqrt{2}$  \_\_\_\_\_

19) If  $a^{2x} = \sqrt{2} - 1$ , then the value of  $\frac{a^{3x} + a^{-3x}}{a^x + a^{-x}}$  is :

19)  $A$  \_\_\_\_\_

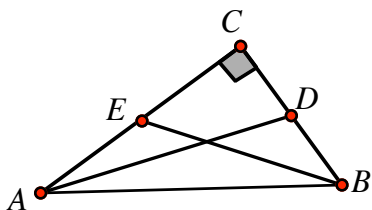
- (a)  $2\sqrt{2} - 1$       (b)  $2 - 2\sqrt{2}$   
(c)  $2\sqrt{2} + 1$       (d)  $\sqrt{2} + 1$

20) If  $x$  and  $y$  satisfy  $\begin{cases} x - y \geq 0 \\ x + y \leq 1 \\ y \geq -1 \end{cases}$  then what is the smallest value attained by  $2x + y$  ?

20)  $-3$  \_\_\_\_\_



- 11) Find all real solutions of  $\sin^2 x + \frac{\sin x}{2} - \frac{1}{2} = 0$  on  $[0, 2\pi]$ . 11)  $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$
- 12) What is the coefficient of  $x^3$  in the expression  $(x^2 + 1)(x - 2)^7$ ? 12) 1008
- 13) If a sequence  $\{a_n\}_{n \geq 1}$  with  $a_1 = 1$  has a partial sum  $s_n$ , and a point  $(a_n, a_{n+1})$  is on the line  $x - y + 1 = 0$ , then  $\frac{1}{s_1} + \frac{1}{s_2} + \dots + \frac{1}{s_n}$  is equal to : 13) C
- (a)  $\frac{n(n+1)}{2}$     (b)  $\frac{2}{n(n+1)}$     (c)  $\frac{2n}{n+1}$     (d)  $\frac{n}{2(n+1)}$
- 14) Find all real values  $x$  such that  $2^{\log_4 x} = 64$ . 14) 4096
- 15) Given  $f(n) = \begin{cases} 1, & n = 0 \\ n \cdot f(n-1), & n \text{ a positive integer} \end{cases}$ , then  $f(6) = ?$  15) 6! or 720
- 16) If  $x, y > 0$  and  $\ln 2^x + \ln 8^y = \ln 2$ , then the smallest value attained by  $\frac{1}{x} + \frac{1}{3y}$  is 16) C
- (a) 2    (b)  $2\sqrt{2}$     (c) 4    (d)  $\sqrt{3}$
- 17) Define  $P = \{x \mid x^2 - 4x - 5 < 0\}$  and  $Q = \{x \mid |x| - a \geq 0\}$ . If  $P \cap Q = \emptyset$ , then  $a$  satisfies 17) B
- (a)  $a > 2$     (b)  $a \geq 5$     (c)  $-1 < a < 5$     (d)  $a > 1$
- 18) If  $|x| \leq \frac{\pi}{4}$ , then what is the minimum value of  $f(x) = \cos^2 x + \sin x$ ? 18)  $\frac{1 - \sqrt{2}}{2}$
- 19) If a straight line  $x + y = 1$  does **not** intersect the circle  $x^2 + y^2 - 2ay = 0$ , where  $a > 0$ , then  $a$  is a member of which of the following intervals? 19) A
- (a)  $(0, \sqrt{2} - 1)$     (b)  $(\sqrt{2} - 1, \sqrt{2} + 1)$   
(c)  $(-\sqrt{2} - 1, \sqrt{2} - 1)$     (d)  $(0, \sqrt{2} + 1)$
- 20) In triangle  $ABC$  given below,  $\angle ACB = 90^\circ$ ,  $D$  and  $E$  are midpoints of  $\overline{BC}$  and  $\overline{AC}$  respectively. If  $\overline{BE} = 4$  and  $\overline{AD} = 7$ , then what is  $\overline{AB}$ ? 20)  $2\sqrt{13}$



# LUZERNE COUNTY MATHEMATICS CONTEST

Luzerne County Council of Teachers of Mathematics

Wilkes University - 2010 Senior Examination

(Section I)

NAME: \_\_\_\_\_

Address: \_\_\_\_\_

SCHOOL: \_\_\_\_\_

City/ZIP: \_\_\_\_\_

Telephone: \_\_\_\_\_

**Directions:** For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

- 1) What is the equation of the line, in slope-intercept form, that is perpendicular to  $2x - 3y = 8$  and passes through  $(1, 4)$ ? 1)  $y = -\frac{3}{2}x + \frac{11}{2}$
- 2)  $|3 - \pi| =$  2) \_\_\_\_\_  $B$   
(a)  $3 - \pi$       (b)  $\pi - 3$       (c)  $-3 - \pi$       (d)  $3 + \pi$
- 3) Suppose the area of a circle is to be doubled. This means the radius of the original circle must be increased by a factor of 3) \_\_\_\_\_  $C$   
(a) 2      (b)  $\frac{1}{2}$       (c)  $\sqrt{2}$       (d)  $\frac{\sqrt{2}}{2}$
- 4)  $(a, b)$  and  $(b, a)$  are points in the Cartesian plane such that  $a \neq b$ . If the distance between  $(a, b)$  and  $(b, a)$  is 5 units, then what is  $|a - b|$ ? 4) \_\_\_\_\_  $\frac{5\sqrt{2}}{2}$
- 5) Find all real numbers  $x$  such that the slope of the line segment passing through the points  $(x^2, 7)$  and  $(-2, 3)$  is  $\frac{1}{8}$ . 5) \_\_\_\_\_  $-\sqrt{30}, \sqrt{30}$
- 6) If a straight line  $x - y + 6 = 0$  is perpendicular to a straight line  $(a + 2)x - y + 1 = 0$ , then what is the value of  $a$ ? 6)  $a = -3$
- 7) If a set  $A = \{x \mid ax^2 - 3x + 2 = 0\}$  has at most one element, then what is the range of  $a$ ? 7) \_\_\_\_\_  $a \geq \frac{9}{8}$  or  $a = 0$
- 8) Find all real solutions to  $e^{\log_8 x} = 50$ . 8) \_\_\_\_\_  $e^{(\ln 50)(\ln 8)}$  or  $50^{\ln 8}$  or  $8^{\ln 50}$
- 9) If  $9a^2 + ka + \frac{1}{9}$  is a complete square, then  $k$  equals: 9) \_\_\_\_\_  $C$   
(a) 2      (b) -2      (c)  $\pm 2$       (d)  $\pm 1$
- 10) What is the remainder when  $x^4 + 20x^3 - x + 5$  is divided by  $x + 5$ ? 10) \_\_\_\_\_  $-1865$

(OVER)

11) In the expression  $\left(\frac{1}{x} + x^2\right)^6$ , what is the coefficient of  $x^3$ ? 11) \_\_\_\_\_ 20 \_\_\_\_\_

12) Given a function  $f(x) = x^3 + \sin x + 1, x \in \mathbb{R}$ . If  $f(a) = 2$ , then what is  $f(-a)$ ? 12) \_\_\_\_\_ B \_\_\_\_\_  
 (a) 3 (b) 0 (c) -1 (d) -2

13) If  $f(\log_2 x) = 2^x$ , then  $f(3)$  equals 13) \_\_\_\_\_ B \_\_\_\_\_  
 (a) 128 (b) 256 (c) 512 (d) 8

14) Define a function  $f(x) = \begin{cases} x + 2, & x \leq -1 \\ x^2, & 1 < x < 2 \\ 2x, & x \geq 2 \end{cases}$ . If  $f(x) = 3$ , then what is the value of  $x$ ? 14) \_\_\_\_\_  $\sqrt{3}$  \_\_\_\_\_

15) Find all real solutions to  $\cos(\arcsin x) = x + 1$ . 15) \_\_\_\_\_ -1, 0 \_\_\_\_\_

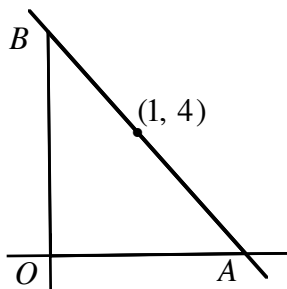
16) If  $(x^2 + y^2)(x^2 + y^2 - 2) + 1 = 0$ , then what is the value of  $x^2 + y^2$ ? 16) \_\_\_\_\_ 1 \_\_\_\_\_

17) What is the greatest common divisor of 2016 and 384? 17) \_\_\_\_\_ 96 \_\_\_\_\_

18) If  $a = \sqrt{2} - 2$ , what is the value of  $\frac{4}{a^2 - 4} + \frac{2}{a + 2} - \frac{1}{a - 2}$ ? 18) \_\_\_\_\_  $\frac{\sqrt{2}}{2}$  \_\_\_\_\_

19) What is the minimum value attained by  $f(x) = \log_2\left(x + \frac{1}{x - 1} + 5\right)$ ? 19) **discarded!** See if you \_\_\_\_\_ can find out why \_\_\_\_\_

20) The straight line  $y = kx + b$  passes through a point  $(1, 4)$ , intersects the  $x$ -axis at  $A$ , and the  $y$ -axis at  $B$ , as shown in the figure below.  $O$  is the origin. If  $\triangle AOB$  has minimal area, then 20) \_\_\_\_\_ A \_\_\_\_\_



- (a)  $k = -4, b = 8$
- (b)  $k = -4, b = 4$
- (c)  $k = -2, b = 4$
- (d)  $k = -2, b = 2$

# LUZERNE COUNTY MATHEMATICS CONTEST

Luzerne County Council of Teachers of Mathematics  
Wilkes University - 2010 Senior Examination  
(Section II)

NAME: \_\_\_\_\_ Address: \_\_\_\_\_  
SCHOOL: \_\_\_\_\_ City/ZIP: \_\_\_\_\_  
Telephone: \_\_\_\_\_

**Directions:** For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

- 1) If  $ab = 2$  and  $c = 18a$ , then what is  $bc$ ? 1) \_\_\_\_\_ 36 \_\_\_\_\_
- 2) If the sum of 2 consecutive integers is 113, what is the larger of the two integers? 2) \_\_\_\_\_ 57 \_\_\_\_\_
- 3) If the area of an equilateral triangle is  $16\sqrt{3}$  square units, then what is the length of a side of the triangle? 3) \_\_\_\_\_ 8 \_\_\_\_\_
- 4) If a substance loses half of its mass every 4 years, how much of a 100g sample of the substance will be left after 28 years? 4) \_\_\_\_\_ 0.78125 g \_\_\_\_\_
- 5) If  $f(x) = \begin{cases} e^x, & x \leq 0 \\ \ln x, & x > 0 \end{cases}$ , what is  $f\left(f\left(\frac{1}{2}\right)\right)$ ? 5) \_\_\_\_\_  $\frac{1}{2}$  \_\_\_\_\_
- 6) Express  $0.\overline{123}$  as a common fraction in lowest terms. 6) \_\_\_\_\_  $\frac{41}{333}$  \_\_\_\_\_
- 7) Write  $\frac{2 - 4i}{3 + 7i}$  in the form  $a + bi$ . 7) \_\_\_\_\_  $-\frac{11}{29} - \frac{13}{29}i$  \_\_\_\_\_
- 8) If  $y = kx$  and  $k \neq 0$ , then 8) \_\_\_\_\_ A \_\_\_\_\_
  - (a)  $y$  is directly proportional to  $x$ .
  - (b)  $y$  is inversely proportional to  $x$ .
  - (c) there is not enough information provided to determine if  $y$  is directly or inversely proportional to  $x$ .
- 9) During a sale, a store reduces the price of an item by 20%. By what percentage must the sale price be increased in order to obtain the original price of the item? 9) \_\_\_\_\_ 25% \_\_\_\_\_
- 10) Which of the following functions is equivalent to  $y = x$ ? 10) \_\_\_\_\_ B \_\_\_\_\_
  - (a)  $y = (\sqrt{x})^2$
  - (b)  $y = \sqrt[3]{x^3}$
  - (c)  $y = \sqrt{x^2}$
  - (d)  $y = \frac{x^2}{x}$

(OVER)

11) Find the domain of  $f(x) = \frac{3x}{\sqrt{1-x}} + \ln(3x+1)$ .

11)  $\left(-\frac{1}{3}, 1\right)$

12) How many subsets are there for the set  $A = \{1, 3, 4, 5\}$ ?

12) 16

13) If  $f(x) = (m-1)x^2 + (m-2)x + m^2 - 7m + 12$  is even, then the value of  $m$  is

13) B

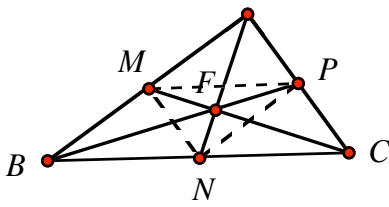
- (a) 1      (b) 2      (c) 3      (d) 4

14) If the real solutions for  $x$  in the equation  $\frac{m}{x+2} = 1$  are negative, then what is the range of  $m$ ?

14)  $m < 2$  and  $m \neq 0$

15) If  $M, N,$  and  $P$  are midpoints of  $\triangle ABC$ ,  $\overline{BF}$  equals:

15) A



- (a)  $2 \cdot \overline{FP}$   
 (b)  $3 \cdot \overline{FP}$   
 (c)  $\overline{FP}$   
 (d)  $\frac{1}{2} \cdot \overline{FP}$

16) Find all real  $x$  such that  $\left(\frac{1}{3}\right)^{x^2-8} > 3^{-2x}$ .

16)  $-2 < x < 4$

17) Five different letters are chosen from *EQUATION* and are arranged in a sequence. How many arrangements can be made with *Q* and *U* next to each other?

17) 960

18) If  $\tan \theta = \sqrt{2}$ , then what is the value of  $\sin^2 \theta - \sin \theta \cos \theta + 2 \cos^2 \theta$ ?

18)  $\frac{4 - \sqrt{2}}{3}$

19) If  $\sin\left(x + \frac{\pi}{4}\right) = -\frac{5}{13}$ , then the value of  $\sin 2x$  is equal to:

19) D

- (a)  $\frac{120}{169}$   
 (b)  $\frac{119}{169}$   
 (c)  $\frac{-120}{169}$   
 (d)  $\frac{-119}{169}$

20) If  $a + b = 2$  where  $a$  and  $b$  are real, what is the smallest value attained by  $3^a + 3^b$ ?

20) 6