LUZERNE COUNTY MATHEMATICS CONTEST

Luzerne County Council of Teachers of Mathematics
Wilkes University - 2009 Junior Examination
(Section I)

NAME: ______________________________________ **Address:** ________________________________

SCHOOL: ____________________________________ **City/ZIP:** ________________________________

Telephone: ________________________________

Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) Suppose that a square has sides of length *s* units. If each side is increased by 1.5 units, express the increase, *I*, in the area of the square in terms of *s*.

1) $3s + 2.25$

2) What is the area of the triangle with vertices (-4, -2), (-4, 8), and (6, -2)?

2) 50 units

3) What is the sum of all the prime numbers between 1 and 37 inclusive?

3) 197

4) The harmonic mean of two numbers is the reciprocal of the average of the reciprocals of the two numbers. Find the harmonic mean of 7 and 9.

4) $\frac{63}{8}$

5) There are 14 juniors and 23 seniors in the service club. The club is to send 4 representatives to the state conference. If the members of the club decide to send 2 juniors and 2 seniors, then how many groupings are possible?

5) 23023

6) If $x\log_3 2 = 1$, then 2^x = _______ .

6) 3

7) Find all real numbers x such that the distance between (3, 4) and (x, 2) is equal to 5 units.

7) $3 \pm \sqrt{21}$

8) If the straight line $x + my + 1 = 0$ is perpendicular to the straight line $m^2x - 2y - 1 = 0$, then m is

(a) $\sqrt[3]{2}$ **(b)** 0 or 2 **(c)** 2 **(d)** 0 or $\sqrt[3]{2}$

8) (b)

9) Express the complex number $\left(\frac{2i}{1+i}\right)^2$ in the form $a + bi$ where a and b are real numbers.

9) $2i$

10) Re-express $\frac{10}{\sqrt[3]{5}}$ by rationalizing the denominator.

10) $2\sqrt[3]{25}$

(OVER)

11) If $f(x) = x^2 - 2ax + 2$ and $f(x) \geq a$ when $x \in [-1, \infty)$, then a satisfies which of the following?

(a) $-1 < a < 1$ **(b)** $-2 \leq a \leq 1$

(c) $-3 \leq a \leq -2$ **(d)** $-3 \leq a \leq 1$

11) (d)

12) Find all real values x such that $9^x - 6 \cdot 3^x - 7 = 0$.

12) $\log_3 7$

13) Albert had a bag of apples. He gave $\frac{1}{4}$ of the apples to Sara. He then gave $\frac{1}{3}$ of what was left to Gerald. After giving $\frac{1}{2}$ of the remaining apples to his sister and eating one apple, Albert had 12 apples left. How many apples were originally in the bag?

13) 52

14) What is the range of $f(x) = \dfrac{2}{14 + e^x} + 3$? Express your answer in interval notation.

14) $(3, \frac{22}{7})$ or $(3, 3\frac{1}{7})$

15) If $\sin\theta + \cos\theta = \frac{1}{5}$, then $\sin 2\theta =$ ______ .

15) $-\frac{24}{25}$

16) The constant term in the expression $\left(\sqrt{x} - \frac{1}{x}\right)^9$ is

(a) -36 **(b)** 36 **(c)** -84 **(d)** 84

16) (c)

17) If $a_n = \dfrac{1}{(n+1)(n+2)}$, then the partial sum s_n of the sequence $\{a_n\}_{n \geq 1}$ is

(a) $\frac{1}{2}$ **(b)** $\frac{1}{2} - \frac{1}{n+2}$ **(c)** $\frac{1}{2n+3}$ **(d)** $\dfrac{2n+3}{(n+1)(n+2)}$

17) (b)

18) Let $A = \{x \mid 0 < x < 9,\ x \text{ is a prime}\}$ and $B = \{x \mid 0 < x < 9,\ x \in \mathbb{N}\}$, then the number of sets S satisfying $A \subset S \subseteq B$ is ________ .

18) 15

19) If $a = \dfrac{1}{2 - \sqrt{5}}$, $b = \dfrac{1}{2 + \sqrt{5}}$ then $a + b + ab$ is equal to

(a) $1 + 2\sqrt{5}$ **(b)** $1 - 2\sqrt{5}$ **(c)** -5 **(d)** 3

19) (c)

20) Find the domain of $\sqrt{\dfrac{x^2 + 4x + 3}{x - 5}}$.

20) $[-3, -1] \cup (5, \infty)$

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) Find the area of the circle that passes through (6, 4) and has as its center (2, 3). **1)** 17π

2) Express $i^{27} + 5i + (7+2i)(5-2i)$ as a real number. **2)** 39

3) If the length of the side of a square increases by 50%, then the area of the square increases by
(a) 100% **(b)** 50% **(c)** 300% **(d)** 125% **3)** (d)

4) Find all real values x such that $3^{\frac{6}{\log_7 x}} = \frac{1}{27}$. **4)** $\frac{1}{49}$

5) Three digits are chosen from 1, 2, 3, 4, 5 at one time. What is the probability that two odd digits remain? **5)** $\frac{3}{10}$

6) If $f(x)$ satisfies $f(x+2) = \frac{13}{f(x)}$ and $f(1) = 2$, then $f(99)$ is equal to _____ . **6)** $\frac{13}{2}$

7) How many 3-element subsets does the set $\{1, 2, 3, 4, 5, 6, 7\}$ have? **7)** 35

8) Find all solutions to $\tan^4 2x - 9 = 0$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. **8)** $-\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{3}$

9) Find all real solutions to the equation $\frac{x^2 - 2}{x} + \frac{2x}{x^2 - 2} = 3$. **9)** $2, -1, 1 \pm \sqrt{3}$

10) If $x^2 + mx - 15 = (x+3)(x+n)$, then $n =$
(a) -5 **(b)** 5 **(c)** -2 **(d)** 2 **10)** (a)

(OVER)

11) Find all real values of k such that $x = -4$ is a root of $P(x) = kx^2 + kx + 3$.

11) $-\frac{1}{4}$

12) The equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$ represents

(a) a circle with radius 5.

(b) an ellipse with eccentricity $\frac{4}{5}$.

(c) a circle with radius 3.

(d) an ellipse with eccentricity $\frac{3}{5}$.

12) (b)

13) Convert $(\sqrt{2}, -\frac{\pi}{4})$ from polar to rectangular coordinates.

13) $(1, -1)$

14) Find the domain of $y = \sqrt{\log_{\frac{2}{5}} x - 1}$.

14) $0 < x \leq \frac{2}{5}$

15) Find all real solutions to the equation $\sqrt{x} - 5\sqrt{\sqrt{x}} - 14 = 0$.

15) 2401

16) Solve for x: $\frac{3}{4} \leq \frac{1}{x-3} < 5$.

16) $\frac{16}{5} < x \leq \frac{13}{3}$

17) A regular polygon having interior angles whose sum is 720° is called

(a) a quadrilateral. **(b)** an octagon. **(c)** a hexagon.
(d) a heptagon. **(e)** none of the above.

17) (c)

18) If $f(x) = \begin{cases} 2e^{x-1}, & x < 2 \\ \log_3(x^2 - 1), & x \geq 2 \end{cases}$, then $f\left(f(2)\right) =$ ______ .

18) 2

19) Let $f(x) = ax^3 + bx + 7$. If $f(5) = 3$, then what is $f(-5)$?

19) 11

20) If m and n satisfy $m + 4\sqrt{mn} - 2\sqrt{m} - 4\sqrt{n} + 4n = 3$,

then $\frac{\sqrt{m} + 2\sqrt{n} - 8}{\sqrt{m} + 2\sqrt{n} + 2002}$ is equal to _______.

20) $-\frac{1}{401}$

LUZERNE COUNTY MATHEMATICS CONTEST

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

1) What is the largest integer less than 2000 that is divisible by 7? **1)** 1995

2) Suppose $A = \sqrt{BC + BD}$. Solve for B in terms of A, C, and D. **2)** $\dfrac{A^2}{C + D}$

3) A square and a circle have equal areas. Express the radius of the circle, r, in terms of the side of the square, s. **3)** $\dfrac{5}{\sqrt{\pi}}$

4) The sum of three consecutive odd numbers is 15. What is the product of these numbers? **4)** 105

5) Compute $\lim_{x \to -4} \dfrac{5x + 20}{x^2 - 16}$. **5)** $-\dfrac{5}{8}$

6) Find all real solutions to $e^x - 8e^{-x} + 2 = 0$. **6)** $\ln 2$

7) Let $f(x) = \dfrac{1 + x}{1 - x}$, $f_1(x) = f(x)$, and $f_{n+1}(x) = f\left(f_n(x)\right)$, then $f_{2006}(x) =$ **7)** (b)

(a) x **(b)** $-\dfrac{1}{x}$ **(c)** $\dfrac{1 + x}{1 - x}$ **(d)** $\dfrac{x - 1}{x + 1}$

8) Find all real numbers x such that $x^2 < 2x + 3$ and $\log_2(x - 1) < 1$. **8)** $1 < x < 3$

9) Express $0.1\overline{245}$ as a common fraction in lowest terms. **9)** $\dfrac{622}{4995}$

10) A rectangular sheet of metal is 20 inches wide. The length of a diagonal between opposite corners is 5 inches more than the length of the whole sheet. What is the length of the sheet of metal? **10)** 37.5 inches

(OVER)

11) How many ways are there to put 14 indistinguishable balls into 3 distinguishable urns if each urn is to contain at least one ball? **11)** 78

12) The number of real solutions to the system of equations **12)** (b)

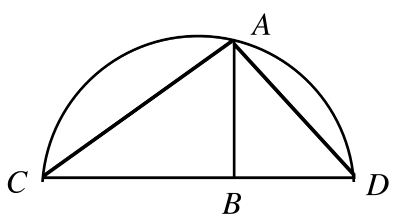
$$\begin{cases} |x| + y = 12 \\ x + |y| = 6 \end{cases}$$ is

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4

13) The remainder of $(x^{100} + 75x - 75B) \div (x - B)$ written in terms of B = _______ . **13)** B^{100}

14) $f(x)$ satisfies $f(x+2) = \frac{1}{f(x)}$. If $f(1) = -5$, then what is $f(-f(5))$? **14)** -5

15) $\overline{CD}$ is the diameter of a semicircle. A is a point on the semicircle, and $\overline{AB}$ is perpendicular to $\overline{CD}$. If $\overline{AB}$ = 20 and $\overline{BD}$ = 10, then $\overline{BC}$ = **15)** (d)



(a) 45 **(b)** 30 **(c)** 15 **(d)** 40

16) Three line segments are randomly selected from five line segments with lengths 1, 3, 5, 7, and 9 (without replacement). What is the probability of forming a triangle using the three chosen segments? **16)** $\frac{3}{10}$

17) If $\sin\alpha - \cos\alpha = -\frac{\sqrt{5}}{2}$, then $\tan\alpha + \frac{1}{\tan\alpha}$ = ______ . **17)** -8

18) Find the minimum value attained by y if $y = \sin^2 x - 3\sin x + 1$. **18)** -11

19) If $a \geq 0, b \geq 0$, and $a + b = 4$, then **19)** (b)

(a) $\frac{1}{ab} \geq \frac{1}{2}$ **(b)** $\frac{1}{a} + \frac{1}{b} \geq 1$

(c) $\sqrt{ab} \geq 2$ **(d)** $\frac{1}{a^2 + b^2} \geq \frac{1}{4}$

20) If $x^2 + y^2 = 1$, then what is the largest value of $3x + 4y$? **20)** 5

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Directions: For each problem, write your answer in the space provided. Do not use approximations. Simplify all fractions and radicals. Your answer must be complete to receive credit for a problem.

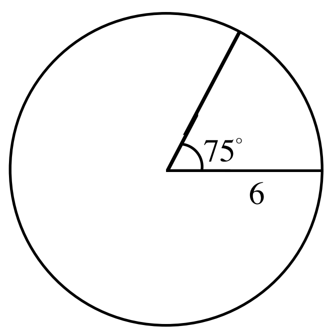
1) What is the greatest common divisor of 34,650 and 2,574? **1)** 198

2) What is the area of a square that has a diagonal length of $8\sqrt{2}$ units? **2)** 64

3) Find the equation of the line, in slope-intercept form, that is the perpendicular bisector of the line segment with endpoints (3, 7) and (-1, 5). **3)** $-2x + 8$

4) A rectangle is twice as long as it is wide. The area of the rectangle is 72 square units. What is the perimeter of the rectangle? **4)** 36

5) Find the area of the sector shown at the right. **5)** $\frac{15\pi}{2}$



6) Find the domain of the function $\sqrt{\log_{0.5}(3x - 2)}$. **6)** $\frac{2}{3} < x \le 1$

7) What is the maximum value of the function $f(x) = 3 - 8x - x^2$? **7)** 19

8) The inverse function of $f(x) = \sqrt{1 + 4x - x^2}$, $x \le 2$ is **8)** (c)

(a) $f(x) = 2 + \sqrt{5 - x^2}$, $0 \le x \le \sqrt{5}$

(b) $f(x) = 2 + \sqrt{5 - x^2}$, $-\sqrt{5} \le x \le 0$

(c) $f(x) = 2 - \sqrt{5 - x^2}$, $0 \le x \le \sqrt{5}$

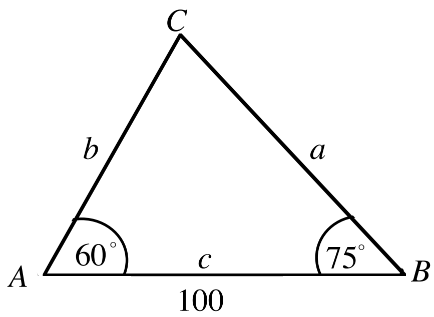
(d) $f(x) = 2 - \sqrt{5 - x^2}$, $-\sqrt{5} \le x \le 0$

9) Find all real values of x such that $5^x = 125^{4x+1}$. **9)** $-\frac{3}{11}$

10) What is the sum of the integers 57 through 88 inclusive? **10)** 2320

(OVER)

11) Find the length of side a in the triangle below. **11)** $50\sqrt{6}$



12) Bob has \$1.55 in change consisting of quarters, dimes, and nickels. If he has 2 more dimes than quarters, and twice as many nickels as quarters, how many nickels does Bob have? **12)** 6

13) If the equation $(m-2)x^2 - 2x + 1 = 0$ has a real solution, then **13)** (b)

(a) $m < 3$ **(b)** $m \le 3$

(c) $m < 3$ and $m \ne 2$ **(d)** $m \le 3$ and $m \ne 2$

14) Find all real values x such that $e^{2x} - 5e^{x} - 24 = 0$. **14)** ln 8

15) If $f(x) = 1 - 2x,\ \ g[f(x)] = \dfrac{1-x^2}{x^2},\ x \ne 0$, what is $g(\frac{1}{2})$? **15)** 15

16) If the sum of all the coefficients in the expansion of $\left(x + \dfrac{1}{x}\right)^n$ is 64, then what is the value of the constant term? **16)** 20

17) If the function $f(x) = \log_a\left(\dfrac{1-mx}{x-1}\right),\ (a > 0,\ a \ne 1,\ m \ne 1)$ is an odd function, then what is the value of m ? **17)** -1

18) If a and b are real numbers, and $a^2 + b^2 = 2$, then **18)** (b)

(a) $a + b < 2$ **(b)** $a + b \le 2$

(c) $a + b > 2$ **(d)** $a + b \ge 2$

19) If $\begin{cases} 2x + 3y \le 6 \\ x - y \ge 0 \\ y \ge 0 \end{cases}$, then what is the largest value of $3x + y$? **19)** 9

20) The even function $f(x)$ is a monotone function on $[0,\ a],\ \ a > 0$. If $f(0) \cdot f(a) < 0$, then what is the number of roots of $f(x)$ on $[-a,\ a]$? **20)** 2