Luzerne County Council of Teachers of Mathematics
Wilkes University – 1998 Senior Examination

(Section I)

Directions: Do not use approxamations. Simplify all fractions and radicals. Your answer must be complete to receive credit for the problem.

1) Determine the domain and range of the function $(f \circ g)(x)$, where

$$f(x) = 3x^{\frac{3}{4}} - 2$$
$$g(x) = x^2 - 2x$$

2) How many *distinct* seven–card hands can be constructed from ten cards labeled 1 through 10 (assume the order of the cards is not relevant)?
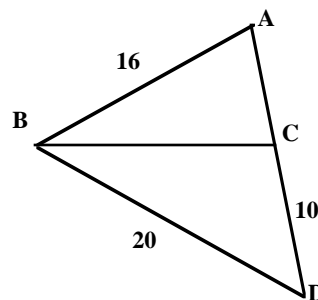
3) Solve the equation $\log_3(x + 3) + \log_3(x - 5) = 2$.

4) Evaluate $\displaystyle\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^3 + 2x^2 - x - 2}$.

5) Express i^i as a real number (where $i^2 = -1$).

6) If one of the solutions of $kx^2 + 3x + k = 0$ is -2, then find the other solutions.

7) In the triangle shown, $m(\angle ABC) = m(\angle DBC)$. If $\overline{AB} = 16$, $\overline{BD} = 20$, and $\overline{CD} = 10$, determine $\overline{AC}$.

8) Solve the following inequality: $\left(\frac{1}{4}\right)^x < \left(\frac{1}{2}\right)^x$.

9) Find all real roots of the polynomial equation
$$4x^4 - 13x^2 + 3 = 0$$

10) A circle is circumscribed about a square which has side length of 4. Another circle is inscribed in the same square. Find the area of the region between the two circles.

11) Find real numbers A and B such that

$$\frac{2x + 1}{x^2 + x - 2} = \frac{A}{x - 1} + \frac{B}{x + 2}$$

12) The area of a circle can be tripled by increasing the radius by 1 unit. Find the radius of the circle.

13) If $F'(x) = \sec^2 x + x$ and $F(0) = 1$, find $F(x)$.

14) How much pure alcohol must be added to 10 liters of a 20% alcohol mixture to obtain a mixture that is 50% alcohol?

15) Represent $0.\overline{617} = 0.617617617\ldots$ as a ratio of two positive integers.

16) Find a real number K so that the line $y = x + K$ is tangent to the parabola with equation $y = 2x^2 - 3x + 4$.

17) In the triangle $\triangle ABC$ shown, $\angle BAC = 60°$, $\angle B = 30°$, $\overline{BC} = 12$, and $\overline{AD}$ bisects $\angle BAC$. Find the length of segment $\overline{DC}$.
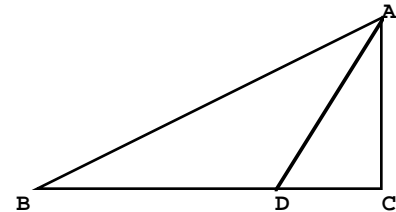
18) Find two consecutive integers, m and n, such that $m < \log_5 80 < n$.

19) Which of the following is *not* true about the graph of $f(t) = 2 \sin(t - \frac{\pi}{12})$?
    a) It has no sharp corners.
    b) It crosses the horizontal axis more than once.
    c) It rises higher and higher as t gets larger.
    d) It is periodic.
    e) It has no vertical asymptotes.

20) Solve the following system of equations for x and y:
$$5x - 3y = -2$$
$$-x + 2y = 3$$

(Section II)

1) Simplify the expression $\dfrac{1 - \dfrac{2}{x+1}}{\dfrac{1}{x} - x}$.

2) A certain fungus grows in a circular shape. Its diameter after t weeks is $6 - \dfrac{50}{t^2 + 10}$ inches. Express the area covered by the fungus as a function of time.

3) Given $f(x) = x^3 + 7x$, compute and simplify $(h \neq 0)$

$$\frac{f(x+h) - f(x)}{h}$$

4) Suppose $f : \mathfrak{R} \to \mathfrak{R}$ is a function such that
   i) $f(xy) = f(x) + f(y)$ for all x, y, in $\mathfrak{R}$
   ii) $f(x/y) = f(x) - f(y)$ for all x, y, in $\mathfrak{R}$
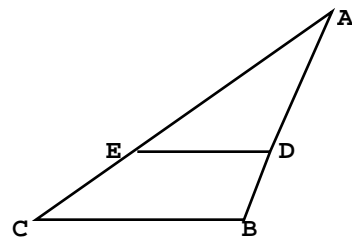
   If $f(2) = 0.69$, $f(3) = 0.80$, and $f(5) = 0.91$, compute $f(\frac{36}{25})$.

5) The sum of five consecutive odd integers is 175. Find the median of the five odd integers.

6) Evaluate $\lim\limits_{x \to \infty} \dfrac{\sin x}{e^x}$.

7) Solve the equation $5 + \sqrt{3x - 11} = x$.

8) In the figure shown, $\overline{DE} \parallel \overline{BC}$, $m(\overline{AD}) = 3$, $m(\overline{BD}) = 2$, and $m(\overline{DE}) = 4$. Find $m(\overline{BC})$.
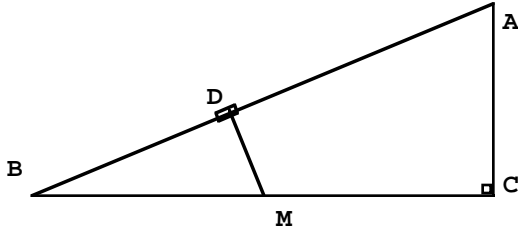
9) What is the domain of the function $f \circ f$ if $f(x) = \log_{10} x$?

10) Find all complex numbers of the form $a + bi$, where a and b are real such that

$$(a + bi)^2 = 3 + 4i$$

11) Express the inequality $\left| x - 6 \right| < 2$ in the form $a < x < b$ for some constants a and b.

12) Find the domain of the function $f(x) = \dfrac{x^2 - x - 2}{x^3 + x^2 - x + 1}$.

13) How many ways are there to place 12 identical balls into 4 distinct urns so that each urn is not empty?

14) Find all real numbers k such that $kx^2 - 2kx + 4 = 0$ has **no real roots.**

15) What is the coefficient of $x^{17}y^3$ in the expansion of $(x + 2y)^{20}$?

16) Which number posses the property that the square of the number is less than the sum of five times this number and 4?

17) The shortest side of a triangle has length 3. Find the shortest side of a similar triangle whose area is twice that of the first triangle.

18) In the following picture, ΔABC is a right triangle with right angle at C. If $\overline{AC} = 5$, $\overline{BC} = 12$, M is the midpoint of $\overline{BC}$, and $\overline{MD} \perp \overline{AB}$, then find the length of $\overline{MD}$.

19) The arithmetic mean of two numbers a and b is 10 and their geometric mean is 8. Find a and b.

20) If $\log_2(\log_c 9) = -1$, then find c.